(1) Here we take the functional that gives the area

$$F[y(x)] = \int_{-a}^{a} y\,dx \qquad y(-a) = y(a) = 0$$

and subject it to a constraint

$$G[y(x)] = \int_{-a}^{a} \sqrt{1+y'^2}\,dx = \ell$$

The extrema can be found using the method of Lagrange multipliers. We form the functional

$$H[y] = F[y] + \lambda G[y] = \int_{-a}^{a} \left(y + \lambda\sqrt{1+y'^2}\right) dx$$

The corresponding Euler-Lagrange equation is

$$\frac{d}{dx}\frac{\partial H}{\partial y'} - \frac{\partial H}{\partial y} = 0 \qquad \Longrightarrow \lambda \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} + 1 = 0$$

that yields

$$\lambda \frac{y'}{\sqrt{1+y'^2}} + x = C_1 \qquad (C_1 = \text{const})$$

or

$$\lambda^2 y'^2 = (x - C_1)^2 (1+y'^2)$$

$$y'^2 (\lambda^2 - (x - C_1)^2) = (x - C_1)^2$$

$$y' = \frac{x - C_1}{\sqrt{\lambda^2 - (x-C_1)^2}}$$

Integrating the last expression gives

$$y(x) = -\sqrt{\lambda^2 - (x-C_1)^2} + C_2 \qquad (C_2 = \text{const})$$

which can be expressed as an equation that describes a circle:

$$(x - C_1)^2 + (y - C_2)^2 = \lambda^2$$

where $C_1 = 0$ because $y(a) = y(-a)$;

② The spherical symmetry of the potential makes it obvious that the most suitable set of coordinates are the spherical ones. The kinetic energy in the spherical coordinates r, θ, ϕ is given by

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2\right)$$

while the Lagrangian is

$$L = T - V = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2\right) + \frac{k}{r}$$

a) First let us determine the expressions for generalized momenta : $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ $\quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ $\quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta \, \dot{\phi}$

The Hamiltonian is then

$$H = \sum_i p_i \dot{q}_i - L = p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) - \frac{k}{r}$$

b) The Hamilton equations of motion are

$$\frac{p_r}{m} = \dot{r} \qquad \frac{p_\theta}{mr^2} = \dot{\theta} \qquad \frac{p_\phi}{mr^2 \sin^2\theta} = \dot{\phi}$$

$$-\frac{k}{r^2} + \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3 \sin^2\theta} = \dot{p}_r \qquad \frac{p_\phi^2 \cot\theta}{mr^2 \sin^2\theta} = \dot{p}_\theta \qquad 0 = \dot{p}_\phi$$

c) ϕ is a cyclic coordinate. Because ϕ is cyclis the corresponding generalized momentum is an integral of motion, i.e. $p_\phi = \text{const}$

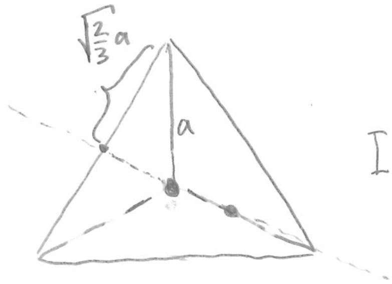
(3) CH_4 molecule has its center of mass located at the C atom. This is intuitively obvious from the symmetry.

This molecule is a spherical **top** – its moment of inertia about any axis that passes through the C atom is the same. This becomes obvious if we realize that there are 4 independent 3D rotations that transform the molecule to itself. We know that the moment of inertia about an axis defined by a unit vector $\hat{n} = (n_1, n_2, n_3)$ where $n_1^2 + n_2^2 + n_3^2 = 1$ is given by

$$I_{\hat{n}} = \hat{n}^T I \hat{n} = I_{11} n_1^2 + I_{22} n_2^2 + I_{33} n_3^2 + 2 I_{12} n_1 n_2 + 2 I_{13} n_1 n_3 + 2 I_{23} n_2 n_3$$

The only way for $I_{\hat{n}}$ be the same for 4 independent sets of $\hat{n}$ (all non-planar) is that $I_{11} = I_{22} = I_{33}$ $I_{12} = I_{13} = I_{23} = 0$

Now once we establish that CH_4 is a spherical top, the easiest way to compute the moment of inertia is to pick the axis that goes throug the middle of two opposite edges (the edges will be perpendicular to the axis)



$$I_{11} = I_{22} = I_{33} = 4 \cdot m \left(\sqrt{\tfrac{2}{3}}a\right)^2 = \frac{8}{3} m a^2$$

(4) The kinetic energy of the three coupled pendula is

$$T = \frac{1}{2} m l^2 (\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2)$$ where l is the pendulum length and m is its mass

The height of each pendulum is

$$y_i = l(1-\cos\phi_i) \approx \frac{l}{2}\phi_i^2$$

So the gravitational potential energy is

$$V_g = \frac{1}{2} mgl(\phi_1^2 + \phi_2^2 + \phi_3^2)$$

The spring potential energy is

$$V_s = \frac{1}{2} k l^2 [(\phi_2-\phi_1)^2 + (\phi_3-\phi_2)^2] = \frac{1}{2} k l^2 (\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_2\phi_3)$$

With that our full Lagrangian is

$$L = \frac{1}{2} m l^2 (\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2) - \frac{1}{2} m l^2 \left[\omega_g^2(\phi_1^2 + \phi_2^2 + \phi_3^2) + \omega_s^2(\phi_1^2 + 2\phi_2^2 + \phi_3^2 - 2\phi_1\phi_2 - 2\phi_2\phi_3)\right]$$

where $\omega_g = \sqrt{\frac{g}{l}}$ $\omega_s = \sqrt{\frac{k}{m}}$

The equations of motion are

$$\ddot{\phi}_1 + \omega_g^2\phi_1 + \omega_s^2\phi_1 - \omega_s^2\phi_2 = 0$$

$$\ddot{\phi}_2 + \omega_g^2\phi_2 + 2\omega_s^2\phi_2 - \omega_s^2\phi_1 - \omega_s^2\phi_3 = 0$$

$$\ddot{\phi}_3 + \omega_g^2\phi_3 + \omega_s^2\phi_3 - \omega_s^2\phi_2 = 0$$

or, in matrix form,

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{M} \begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\phi}_3 \end{pmatrix} = - \underbrace{\begin{pmatrix} \omega_g^2+\omega_s^2 & -\omega_s^2 & 0 \\ -\omega_s^2 & \omega_g^2+2\omega_s^2 & -\omega_s^2 \\ 0 & -\omega_s^2 & \omega_g^2+\omega_s^2 \end{pmatrix}}_{K} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

For small oscillations the solutions have the form

$$\vec{\phi}(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \\ \phi_3(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{i\omega t}$$

in which case the above matrix equation yields

$$(K - \omega^2 M)\vec{a} = 0$$

$$K - \omega^2 M = \begin{pmatrix} \omega_g^2 + \omega_s^2 - \omega^2 & -\omega_s^2 & 0 \\ -\omega_s^2 & \omega_g^2 + 2\omega_s^2 - \omega^2 & -\omega_s^2 \\ 0 & -\omega_s^2 & \omega_g^2 + \omega_s^2 - \omega^2 \end{pmatrix}$$

Setting the determinant to zero leads to

$$(\omega_g^2 - \omega^2)(\omega_g^2 - \omega_s^2 - \omega^2)(\omega_g^2 + 3\omega_s^2 - \omega^2) = 0$$

with solutions

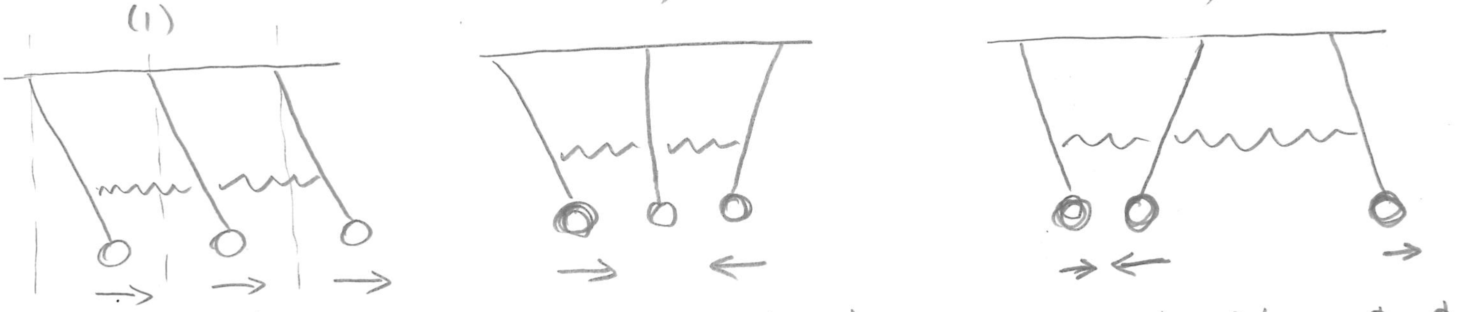
$$\omega_1^2 = \omega_g^2 = \frac{g}{e} \qquad \omega_2^2 = \omega_g^2 + \omega_s^2 = \frac{g}{e} + \frac{k}{m} \qquad \omega_3^2 = \omega_g^2 + 3\omega_s^2 = \frac{g}{e} + 3\frac{k}{m}$$

The corresponding eigenvectors (I skip basic algebra) are

$$\vec{a}^{(1)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{a}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \vec{a}^{(3)} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

These three eigenvectors allow to depict the three normal modes graphically

(1) (2) (2)



$\phi_1 = \phi_2 = \phi_3$ $\qquad$ $\phi_1 \quad \phi_2 = 0 \quad \phi_3 = -\phi_1$ $\qquad$ $\phi_1 \quad \phi_2 = -2\phi_1 \quad \phi_3 = \phi_1$

(5) Let K be the laboratory frame. Let K' be the frame moving along the x-axis with velocity v (particle 1 is at rest in this frame). Let K'' be the frame moving along the y-axis with velocity v (particle 2 is at rest in this frame).

The transformation of particle 1 coordinates and time from K' to K is ($\beta \equiv \frac{v}{c}$ $\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}$):

$$\left.\begin{aligned} x &= \gamma(x'+vt') \\ y &= y' \\ z &= z' \\ t &= \gamma(t'+\tfrac{\beta}{c}x') \end{aligned}\right\} \quad (1)$$

The transformation of particle 1 coordinates and time from K to K'' is

$$\left.\begin{aligned} x'' &= x \\ y'' &= \gamma(y-vt) \\ z'' &= z \\ t'' &= \gamma(t-\tfrac{\beta}{c}y) \end{aligned}\right\} \quad (2)$$

Substituting (1) into (2) we obtain

$$x'' = \gamma x' + \gamma v t'$$

$$y'' = \gamma y' - \gamma^2 v t' - \gamma^2 v \frac{\beta}{c} x'$$

$$z'' = z'$$

$$t'' = \gamma^2 t' + \gamma^2 \frac{\beta}{c} x' - \gamma \frac{\beta}{c} y'$$

From the last equation we have $t' = \frac{t''}{\gamma^2} - \frac{\beta}{c}x' + \frac{\beta}{\gamma c}y'$.

Then
$$x'' = \gamma x' + \gamma v\left(\frac{t''}{\gamma^2} - \frac{\beta}{c}x' + \frac{\beta}{\gamma c}y'\right)$$

$$y'' = \gamma y' - \gamma^2 v\left(\frac{t''}{\gamma^2} - \frac{\beta}{c}x' + \frac{\beta}{\gamma c}y'\right)$$

$$z'' = z'$$

The components of the relative velocity (velocity of particle 1 relative to particle 2) are

$$u''_x = \frac{dx''}{dt''} = \frac{v}{\gamma}$$

$$u''_y = \frac{dy''}{dt''} = -v$$

$$u''_z = \frac{dz''}{dt''} = 0$$

The absolute value of u'' is

$$u'' = \sqrt{u''^2_x + u''^2_y + u''^2_z} = \sqrt{1 + \frac{1}{\gamma^2}}\, v = \sqrt{2-\beta^2}\, v$$