1. The variation of J is

$$\delta J = J[f(\vec{r}) + \varepsilon\eta(\vec{r})] - J[f(\vec{r})] =$$

$$= \iint \frac{\{f(\vec{r}_1)+\varepsilon\eta(\vec{r}_1)\}\{f(\vec{r}_2)+\varepsilon\eta(\vec{r}_2)\}}{|\vec{r}_1-\vec{r}_2|} d\vec{r}_1 d\vec{r}_2 - \iint \frac{f(\vec{r}_1) f(\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|} d\vec{r}_1 d\vec{r}_2 =$$

$$= \varepsilon \iint \frac{f(\vec{r}_1)\eta(\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|} d\vec{r}_1 d\vec{r}_2 + \varepsilon \iint \frac{f(\vec{r}_2)\eta(\vec{r}_1)}{|\vec{r}_1-\vec{r}_2|} d\vec{r}_1 d\vec{r}_2 + O(\varepsilon^2) =$$

$$= \varepsilon \int \left(\int \frac{f(\vec{r}_1)}{|\vec{r}_1-\vec{r}_2|} d\vec{r}_2 \right) \eta(\vec{r}_2) d\vec{r}_2 + \varepsilon \int \left(\int \frac{f(\vec{r}_1)}{|\vec{r}_2-\vec{r}_1|} d\vec{r}_1 \right) \eta(\vec{r}_2) d\vec{r}_2 + O(\varepsilon^2) =$$

$$= \varepsilon \int \underbrace{\left(2 \int \frac{f(\vec{r}_1)}{|\vec{r}_1-\vec{r}_2|} d\vec{r}_1 \right)}_{\frac{\delta J}{\delta f(\vec{r}_2)}} \eta(\vec{r}_2) d\vec{r}_2 + O(\varepsilon^2)$$

So the first functional derivative is given by:

$$\frac{\delta J}{\delta f(\vec{r})} = 2 \int \frac{f(\vec{r}_1)}{|\vec{r}_1-\vec{r}|} d\vec{r}_1$$

② First functional derivative :

$$\delta F = F[f+\varepsilon\eta] - F[f] = \int_{x_1}^{x_2} \left\{ \left(f(x)+\varepsilon\eta(x)\right)^{5/3} - f^{5/3}(x) \right\} dx =$$

$$= \int_{x_1}^{x_2} f^{5/3} \left\{ \left(1+\frac{\varepsilon\eta}{f}\right)^{5/3} - 1 \right\} dx = \int_{x_1}^{x_2} \frac{5}{3} f^{2/3}(x)\, \varepsilon\eta(x)\, dx + O(\varepsilon^2)$$

$$\frac{\delta F}{\delta f(x)} = \frac{5}{3} f^{2/3}(x) = \int_{x_1}^{x_2} \delta(x-x') \frac{5}{3} f^{2/3}(x')\, dx' \qquad x_1 < x < x_2$$

Second functional derivative :

$$\delta \frac{\delta F}{\delta f(x)} = \int_{x_1}^{x_2} \delta(x-x') \frac{5}{3} \left\{ \left(f(x')+\varepsilon\eta(x')\right)^{2/3} - f^{2/3}(x') \right\} dx' =$$

$$= \int_{x_1}^{x_2} \delta(x-x') \frac{5}{3} f^{2/3}(x') \left\{ \left(1+\frac{\varepsilon\eta(x')}{f(x')}\right)^{2/3} - 1 \right\} dx' =$$

$$= \int_{x}^{x_2} \delta(x-x') \frac{5}{3}\cdot\frac{2}{3} f^{-1/3}(x')\, \varepsilon\eta(x')\, dx'$$

So

$$\frac{\delta^2 F}{\delta f(x')\,\delta f(x)} = \frac{10}{9} \delta(x-x') f^{-1/3}(x')$$

(3) $F[y(x)] = \int_0^1 \left(y'^2(x) + 12xy(x) \right) dx \qquad y(0) = 0 \quad y(1) = 1$

Our integrand here is $f(y, y', x) = y'^2 + 12xy$

The Euler-Lagrange equation is

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}$$

$$\frac{d}{dx} 2y' = 12x$$

or

$$y'' = 6x$$

Then

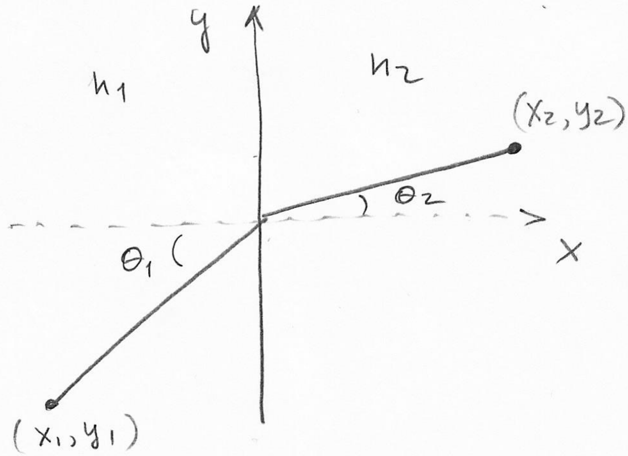
$$y' = 3x^2 + a$$

$$y = x^3 + ax + b \qquad \text{where } a, b \text{ are constants}$$

Applying the boundary condition yields

$$y(x) = x^3$$

(4) First, let us express the travel time t as a functional of path



$$dl^2 = dx^2 + dy^2$$

$$dl = \sqrt{1+y'^2}\, dx$$

The time to travel path dl is

$$dt = \frac{dl}{c/n} = \frac{n}{c}\, dl$$

where c/n is the speed of light in a medium with refraction index n. Then the total travel time is

$$t[y(x)] = \int_{x_1}^{x_2} \frac{n(x)}{c} \sqrt{1+y'^2_{(x)}}\, dx \equiv \int_{x_1}^{x_2} f(y, y', x)\, dx$$

where $n(x)$ can be any function of x. To find the extremum we need to solve the Euler-Lagrange equation

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y} \Rightarrow \frac{d}{dx} \frac{n(x)}{c} \frac{y'}{\sqrt{1+y'^2}} = 0$$

Thus

$$n(x) \frac{y'}{\sqrt{1+y'^2}} = \text{const}$$

Now in our case n is constant in each medium and according to our drawing

$$y'(x) = \tan\theta(x)$$

Therefore $\sqrt{1+y'^2} = \sqrt{1+\tan^2\theta} = \sqrt{\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}} = \frac{1}{\cos\theta}$

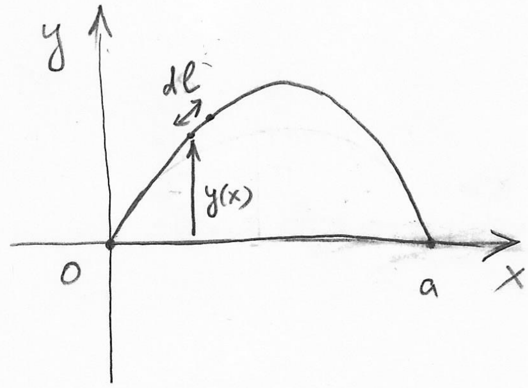
So

$$n(x) \frac{\tan\theta}{\cos\theta} = \text{const} \quad \text{or} \quad n(x) \sin\theta(x) = \text{const}$$

In our case we have only two values of n: n_1 and n_2

Therefore $n_1 \sin\theta_1 = n_2 \sin\theta_2$ or $\frac{n_1}{n_2} = \frac{\sin\theta_2}{\sin\theta_1}$

(5) The volume of a figure of revolution is given by



$$V = \int_0^a \pi y^2 dx \equiv \int_0^a F(y)\, dx$$

where 0 and a are some constants (end points)

The total surface area of revolution is

$$A = \int_0^a 2\pi y\, dl = \int_0^a 2\pi y \sqrt{1+y'^2}\, dx \equiv \int_0^a G(y,y')\, dx = S = \text{const}$$

Hence in this problem we need to maximize functional $V[y(x)]$ subject to the constraint $A[y(x)] = S$

We will use the method of Lagrange multipliers. Let us define

$$H = F + \lambda G = \pi y^2 + \lambda 2\pi y \sqrt{1+y'^2}$$

as our new integrand. As H does not contain x explicitly, we can use the Beltrami equation (which is simpler than the Euler-Lagrange eq.):

$$H - y' \frac{\partial H}{\partial y'} = c \qquad (c = \text{const})$$

$$\pi y^2 + 2\pi\lambda y \sqrt{1+y'^2} - y' \frac{\lambda 2\pi y y'}{\sqrt{1+y'^2}} = c$$

or

$$\pi y^2 + \frac{2\pi\lambda y (1+y'^2) - \lambda 2\pi y y'^2}{\sqrt{1+y'^2}} = c$$

or

$$\pi y^2 + \frac{2\pi\lambda y}{\sqrt{1+y'^2}} = c$$

Now $y(0) = 0$, therefore $c = 0$ and we get

$$y^2 = \frac{-2\lambda y}{\sqrt{1+y'^2}}$$

$$y = \frac{-2\lambda}{\sqrt{1+y'^2}}$$

Then we can solve this for y':

$$y' = \frac{\sqrt{4\lambda^2 - y^2}}{y}$$

The integration gives

$$\int \frac{y\,dy}{\sqrt{4\lambda^2 - y^2}} = \int dx$$

$$-\sqrt{4\lambda^2 - y^2} = x - b \qquad (b = \text{const})$$

or

$$x = b - \sqrt{4\lambda^2 - y^2} \qquad b = 2\lambda$$

When $x = 0$ and $y = 0$

This last equation can be reduced to

$$(x - 2\lambda)^2 + y^2 = 4\lambda^2$$

When $x = a$ $y = 0$, so $(a - 2\lambda)^2 = 4\lambda^2$ and

$a = 4\lambda$

Our equation then becomes

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

which describes a circle of radius $\frac{a}{2}$ centered at $(\frac{a}{2}, 0)$. Hence the figure sought is a sphere.