Hamiltonian mechanics. Hamilton's equations of motion

We started this course with the Newtonian form of mechanics, where the central concepts are forces and accelerations. It was primarily suited in Cartesian coordinate systems. Then we introduced the Lagrangian formulation. It is entirely equivalent to Newton's formulation, but the Lagrangian formulation has two important advantages. It allows to use generalized coordinates (not just the Cartesian ones) $q_1, \dots q_n$ as the Lagrange's equations are equally valid for any choice of $q_1 \dots q_n$. Also, in certain situations it is considerably more tractable. Indeed, sometimes it is not even possible to state explicitly all the forces acting on an object, while it remains relatively simple to give expressions for the kinetic and potential energies. On the other hand the Lagrangian method may have disadvantages when applied to dissipative systems.

Now we want to consider yet another formulation of mechanis – the Hamiltonian mechanics. It is also equivalent to the Newtonian mechanics, but is allows more flexibility in terms of the choice of coordinates. In fact, in that sense it is even more flexible than the Lagrangian mechanics. The central quantity in the Hamiltonian mechanics is the

Hamiltonian function, which in most practical situations is just the total mechanical energy — something that has a clear physical meaning and is often conserved. It is well suited to handle other conserved quantities and provides a natural transition from classical mechanics to quantum mechanics.

The Hamiltonian mechanics rises naturally from the Lagrangian one and so we will start from the Lagrangian, which in most cases is given by

$$L = T - V$$

The Lagrangian is a function of $q_1, \dots q_n$, their time derivatives, $\dot{q}_1, \dots, \dot{q}_n$, and time t. The n coordinates q_n define a point in n-dimensional configuration space, while $2n$ cordinates $q_1, \dots q_n, \dot{q}_1, \dots \dot{q}_n$ define a point in state space

Let us recall the definition of generalized momenta:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

← p_i is the canonical momentum or the momentum conjugate to q_i

The Hamiltonian function (or just Hamiltonia) H is defined as

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L$$

$2n$ coordinates $q_1, \dots q_n, p_1 \dots p_n$ define a point in the phase space

Let us now derive Hamilton's equations of motion for a conservative, one-dimensional system with a single generalized coordinate q

The Lagrangian is

$$L = L(q,\dot{q}) = T(q,\dot{q}) - V(q)$$

For a conservative system V, by definition, depends on q (not $\dot{q}$) only. The kinetic energy always has this **general** form

$$T = \frac{1}{2} A(q)\, \dot{q}^2$$

which can be proven as follows for a general system with n degrees of freedom

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_i m_i \left(\sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j\right)\left(\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k\right) =$$

$$= \sum_{j,k=1}^{n} \underbrace{\left[\sum_i m_i \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\cdot\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)\right]}_{A_{jk}(q_1,\dots q_n)} \dot{q}_j \dot{q}_k$$

According to the definition of the Hamiltonian

$$H = p\dot{q} - L$$

with $p = \frac{\partial L}{\partial \dot{q}} = A(q)\dot{q}$ it becomes

$$H = A(q)\dot{q}^2 - L = 2T - L = T + V$$

Next let us express $\dot{q}$ as $\dot{q} = \frac{p}{A(q)} \equiv \dot{q}(q,p)$. Then

$H = p\dot{q} - L = p\,\dot{q}(q,p) - L(q,\dot{q}(q,p))$ is a function of p and q

In the final step let us evaluate derivatives of H with respect to q and p:

$$\frac{\partial H}{\partial q} = \frac{\partial}{\partial q}\Big(p\,\dot{q}(q,p) - L(q,\dot{q}(q,p))\Big) = p\frac{\partial \dot{q}}{\partial q} - \left[\frac{\partial L}{\partial q} + \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial q}\right] =$$

$$= -\frac{\partial L}{\partial q} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -\frac{d}{dt}p = -\dot{p}$$

$$\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\Big(p\,\dot q(q,p) - L(q,\dot q(q,p))\Big) = \dot q + p\frac{\partial \dot q}{\partial p} - \underbrace{\frac{\partial L}{\partial \dot q}}_{p}\frac{\partial \dot q}{\partial p} =$$

$$= \dot q$$

Therefore we have the following equations of motion for a 1D system:

$$\dot q = \frac{\partial H}{\partial p} \qquad \dot p = -\frac{\partial H}{\partial q}$$

Example: particle restricted to move in 1D with a potential energy $V(x)$

$$L = T - V = \frac{1}{2}m\dot x^2 - V(x)$$

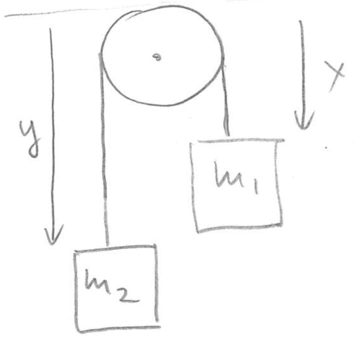
Lagrange equation: $\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot x}$ or $-\frac{\partial V}{\partial x} = m\ddot x$

$$p = \frac{\partial L}{\partial \dot x} = m\dot x \qquad H = p\dot x - L = \frac{p^2}{m} - \left[\frac{p^2}{2m} - V(x)\right] = \frac{p^2}{2m} + V(x)$$

Hamilton's equations:

$$\dot x = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot p = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$$

Example: Atwood's machine



$$L = T - V = \frac{1}{2}(m_1+m_2)\dot x^2 + (m_1-m_2)gx$$

$$p = \frac{\partial T}{\partial \dot x} = (m_1+m_2)\dot x \qquad \dot x = \frac{p}{m_1+m_2}$$

now we can substitute $\dot x$ in H!

$$H = T + V = \frac{p^2}{2(m_1+m_2)} - (m_1-m_2)gx$$

Hamilton's equations:

$$\dot x = \frac{\partial H}{\partial p} = \frac{p}{m_1+m_2} \qquad \dot p = -\frac{\partial H}{\partial x} = (m_1-m_2)g$$

from these equations we find $\ddot x = \frac{m_1-m_2}{m_1+m_2}g$

Now let us generalize the formalism to the case of n degrees of freedom. Assume we have n generalized coordinates:

$$\vec{q} = (q_1, \dots q_n)$$

and generalized velocities and momenta

$$\dot{\vec{q}} = (\dot{q}_1, \dots \dot{q}_n) \qquad \vec{p} = (p_1, \dots p_n)$$

Our starting point is, again, the Lagrangian

$$L = L(\vec{q}, \dot{\vec{q}}, t) = T - V$$

The Hamiltonian is defined as

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \qquad \text{with} \qquad \underbrace{p_i = \frac{\partial L(\vec{q}, \dot{\vec{q}}, t)}{\partial \dot{q}_i} \quad i=1..n}_{\text{these equations can be solved to get } \dot{q}_i(q_1, \dots q_n, p_1 \dots p_n, t)}$$

We can eliminate $\dot{q}_i$ from the Hamiltonian above

$$H = \sum_{i=1}^{n} p_i \dot{q}_i(\vec{q}, \vec{p}, t) - L(\vec{q}, \dot{\vec{q}}(\vec{q}, \vec{p}, t), t)$$

Then we take partial derivatives

$$\frac{\partial H}{\partial q_k} = \frac{\partial}{\partial q_k}\left[\sum_{i=1}^{n} p_i \dot{q}_i(\vec{q}, \vec{p}, t) - L(\vec{q}, \dot{\vec{q}}(\vec{q}, \vec{p}, t), t)\right] =$$

$$= \sum_{i=1}^{n} p_i \frac{\partial \dot{q}_i}{\partial q_k} - \frac{\partial L}{\partial q_k} - \sum_{j=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{p_j} \frac{\partial \dot{q}_j}{\partial q_k} = -\frac{\partial L}{\partial q_k} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} =$$

$$= -\frac{d}{dt} p_k = -\dot{p}_k \implies \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

$$\frac{\partial H}{\partial p_k} = \frac{\partial}{\partial p_k}\left[\sum_{i=1}^{n} p_i \dot{q}_i(\vec{q}, \vec{p}, t) - L(\vec{q}, \dot{\vec{q}}(\vec{q}, \vec{p}, t), t)\right] = \dot{q}_k(\vec{q}, \vec{p}, t) +$$

$$+ \sum_{i=1}^{n} p_i \frac{\partial \dot{q}_i}{\partial p_k} - \sum_{j=1}^{n} \underbrace{\frac{\partial L}{\partial \dot{q}_j}}_{p_j} \frac{\partial \dot{q}_j}{\partial p_k} = \dot{q}_k \implies \dot{q}_k = \frac{\partial H}{\partial p_k}$$

As yet another simple example of the Hamiltonian formalism let us consider a particle in a central force field.

By conservation of angular momentum the motion is confined to a plane. We can define the polar coordinates r and ϕ in this plane. The kinetic energy is then

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right)$$

We must now express T in terms of generalized momenta p_r and p_ϕ. According to the definition

$$p_r \equiv \frac{\partial L}{\partial \dot{r}} = \frac{\partial T}{\partial \dot{r}} = m\dot{r} \qquad \text{(we assume } V = V(r) \text{)}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

p_r is here is just the radial component of $m\vec{v}$ (the ordinary momentum), but p_ϕ is the angular momentum. Now we must solve for $\dot{r}$ and $\dot{\phi}$ in terms of p_r and p_ϕ

$$\dot{r} = \frac{p_r}{m} \qquad \dot{\phi} = \frac{p_\phi}{m r^2}$$

and substitute them into the original expression for T

$$T = \frac{1}{2m} \left(p_r^2 + \frac{p_\phi^2}{r^2} \right)$$

which gives the following Hamiltonian

$$H = T + V = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r)$$

With that we can now write down all four Hamilton equations:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2} \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

The first of these reproduces the definition of the radial momentum. The second (if we substitute the first into it) gives us the familiar result that $m\ddot{r}$ is the sum of the actuar radial force and the centrifugal term $\frac{p_\phi^2}{mr^3}$. The third equation reproduces the definition of p_ϕ. The forth one tells us that the angular momentum is conserved.

This exomple with a particle in a central field illustrates again the general algorithm for setting up the Hamilton equations:

1) Choose generalized coordinates $q_1, .., q_n$
2) Write down the Lagrangian in terms of q_i and $\dot{q}_i$, $i = 1 ... n$.
3) Find the conjugated momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$ $i = 1...n$
4) Solve for $\dot{q}_i$ in terms of p's and q's, $i = 1..n$
5) Write down the Hamiltonian as a function of p's and q's, $H = \sum_i p_i \dot{q}_i - L(\vec{q}, \dot{\vec{q}}(\vec{p}, \vec{q}))$, by replacing all $\dot{q}_i$'s with the expressions obtained in step 4
6) Write down Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$