The Euler angles

In order to describe the orientation of a rigid body in space relative to an observer outside the body (i.e. in the laboratory reference frame) we must specify must specify how the orientation of the body relative to a fixed coordinate system changes in time. While there are many possible ways to do this, a commonly chosen scheme uses three angles — ϕ, θ, ψ — to relate the direction of the principal axes of the body relative to the fixed (lab) frame.

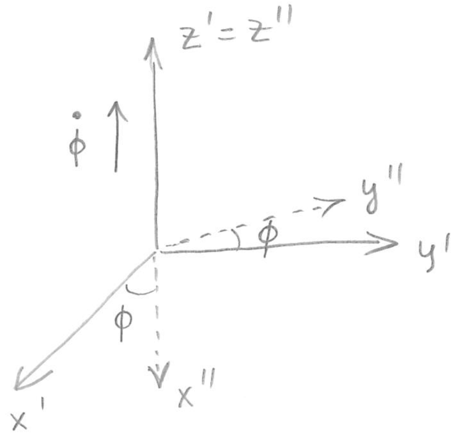
The transformation from one coordinate system to another can be represented in a matrix form as follows:

$$\vec{r} = U\vec{r}\,'.$$

Here $\vec{r}\,'$ denotes the position in the fixed system and $\vec{r}$ denotes the position in the body reference frame. The rotation matrix U, which is an orthogonal matrix (preserves length and angles between vectors it acts on), contains three independent parameters — rotational angles.

The Euler angles are generated in the following series of rotations that take the fixed (lab) system to the body system:

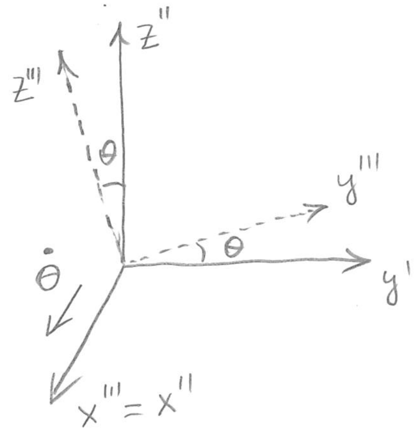
1) The first rotation is counterclockwise through an angle ϕ about the z-axis. Because the rotation takes place in the xy-plane the transformation matrix is



$$U_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\vec{r}\,'' = U_\phi \vec{r}\,'$

2) The second rotation is counterclockwise through an angle θ about the new x'' axis.
The transformation matrix is

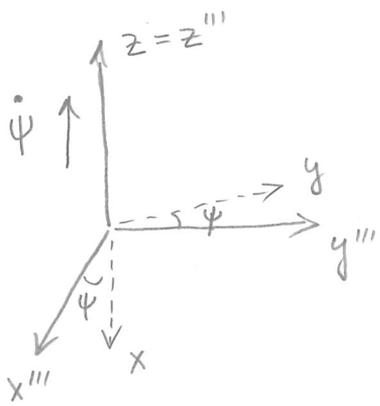


$$U_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

and

$$\vec{r}''' = U_\theta \vec{r}''$$

3) The third rotation is counterclockwise through an angle ψ about the z''' axis. The trasformation matrix is



$$U_\psi = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\vec{r} = U_\psi \vec{r}'''$$

The complete transformation containing all three rotations is given by

$$\vec{r} = U_\psi\left(U_\theta\left(U_\phi \vec{r}'\right)\right) = U_\psi U_\theta U_\phi \vec{r}' = U\vec{r}'$$

where the explicit form of U can be obtained by means of matrix mutiplication:

$$U = \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

We can also find the relation between the time derivatives of the rotation angles

$$\omega_\phi \equiv \dot{\phi} \qquad \omega_\theta \equiv \dot{\theta} \qquad \omega_\psi \equiv \dot{\psi}$$

and the components of the angular velocity $\vec{\omega}$ in the body coordinate system. Note that the angular velocities $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ are directed along the following angles

$\dot{\phi}$ - along z' (fixed/lab frame)

$\dot{\theta}$ - along x''

$\dot{\psi}$ - along z (body frame)

The components of $\dot{\psi}$ in the body frame are trivial:

$$\begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

The components of $\dot{\theta}$ in the body frame are

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} = U_\psi \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{pmatrix}$$

Lastly, the components of $\dot{\phi}$ is the body

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{\phi}_3 \end{pmatrix} = U_\psi U_\theta \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta \\ \dot{\phi}\cos\psi\sin\theta \\ \dot{\phi}\cos\theta \end{pmatrix}$$

By collecting the individual components we then get

$$\omega_1 = \dot{\phi}_1 + \dot{\theta}_1 + \dot{\psi}_1 = \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi$$

$$\omega_2 = \dot{\phi}_2 + \dot{\theta}_2 + \dot{\psi}_2 = \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi$$

$$\omega_3 = \dot{\phi}_3 + \dot{\theta}_3 + \dot{\psi}_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

The Euler equations for a rigid body

Let us consider the torque-free motion of a rigid body. In such a case the potential energy V vanishes and the Lagrangian L becomes identical to the rotational kinetic energy. If we choose the coordinate system that correspond to the principal axes of rotation we have

$$T = \frac{1}{2} \sum_i{}' I_i \omega_i^2 \qquad (*)$$

If we choose the Eulerian angles as the generalized coordinates, then the Lagrange's equation for ψ is

$$\frac{\partial T}{\partial \psi} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = 0$$

It can also be expressed as

$$\sum_i{}' \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \psi} - \frac{d}{dt}\left(\sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \dot{\psi}}\right) = 0 \qquad (**)$$

If we differentiate $\vec{\omega}$ with respect to ψ and $\dot{\psi}$ we set

$$\begin{cases} \dfrac{\partial \omega_1}{\partial \psi} = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi = \omega_2 \\[2ex] \dfrac{\partial \omega_2}{\partial \psi} = -\dot{\phi} \sin\theta \sin\psi - \dot{\theta} \cos\psi = -\omega_1 \\[2ex] \dfrac{\partial \omega_3}{\partial \psi} = 0 \end{cases}$$

and $\dfrac{\partial \omega_1}{\partial \dot{\psi}} = \dfrac{\partial \omega_2}{\partial \dot{\psi}} = 0 \qquad \dfrac{\partial \omega_3}{\partial \dot{\psi}} = 1$

From $(*)$ we also have $\dfrac{\partial T}{\partial \omega_i} = I_i \omega_i$

Then equation $(**)$ becomes

$$I_1\omega_1\omega_2 + I_2\omega_2(-\omega_1) - \frac{d}{dt} I_3\omega_3 = 0$$

or

$$(I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 = 0$$

By permuting indicies 1, 2, 3 we can get relations for $\dot{\omega}_1$ and $\dot{\omega}_2$:

$$\left.\begin{aligned} (I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 &= 0 \\ (I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 &= 0 \end{aligned}\right\} \quad \text{Euler's equation for a torque-free motion}$$

To obtain the Euler equations for the case when torques are present we start with

$$\left(\frac{d\vec{L}}{dt}\right)' = \vec{N} \qquad \text{(prime stands for "fixed" frame)}$$

In a previous lecture we also showed that

$$\left(\frac{d\vec{L}}{dt}\right)' = \left(\frac{d\vec{L}}{dt}\right) + \vec{\omega} \times \vec{L}$$

no prime stands for "body" frame

Hence

$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{N}$$

If we project this equation on the z-axis we get

$$\dot{L}_z + \omega_x L_y - \omega_y L_x = N_z \qquad (***)$$

However, since we have chosen the coordinate system in such a way that its axes coincide with the principal axes of the body we also have

$$L_i = I_3 \omega_i$$

Then equation (***) becomes

$$I \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3$$

We can either repeat this procedure manually and project the $\frac{d\vec{L}}{dt} + \vec{\omega}\times\vec{L} = \vec{N}$ equation on y and x axes, or make thing more general by recalling that

$$\vec{a}\times\vec{b} = \varepsilon_{ijk}\,\vec{e}_i\, a_j b_k$$

where ε_{ijk} is the Levi-Civita symbol and summation over repeated indicies is assumed

so that

$$\dot{L}_i + \varepsilon_{ijk}\,\omega_j L_k = N_i \qquad i = 1,2,3$$

With this the equations of motions are

$$I_i \frac{d\omega_i}{dt} + \varepsilon_{ijk}\,\omega_j \omega_k I_k = N_i \qquad i = 1,2,3$$

Or, we can write them in the expanded form

$$I_1 \dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1$$

$$I_2 \dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2$$

$$I_3 \dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3$$