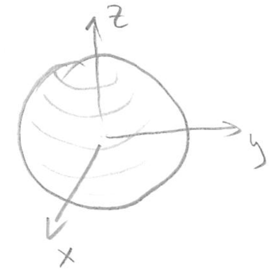
Torque-free motion of a symmetric top

The term "top" is used to denote a rigid, rotating body. A top is called spherical if all three of its principal moments of inertia are equal. A sphere or a uniform cube are examples of spherical tops. A top is called symmetric if two of its moments of inertia are equal.

$I_1 = I_2 = I_3$

spherical top



$I_1 = I_2 > I_3$

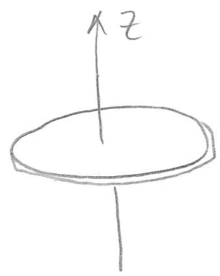
prolate top

long cilinder

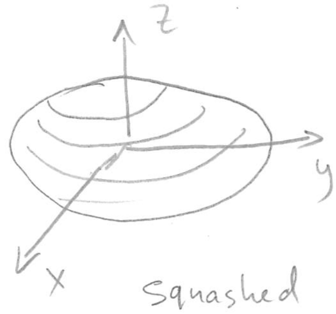
prolonged ellipsoid (cigar-shaped)

$I_1 = I_2 < I_3$

oblate top or flattened top



disk



squashed ellipsoid

Recall that principal moments of inertia correspond to principal axes (which are orthogonal to each other). If two moments of inertia are equal (e.g. two eigen-values of tensor I are degenerate) that means we can choose any linear combination of those two principal axes and the moment of inertia will be the same.

Now let us consider a symmetric top. Its motion is described by the Euler equations. Those three coupled differential equations for $\omega_1(t)$, $\omega_2(t)$, and $\omega_3(t)$ are not linear. In general the solutions $\omega_i(t)$ are complicated. In the case of a torque-free motion, however and when two principal moments of inertia are equal the equations become considerably simpler:

$$\begin{cases} I_1 \dot{\omega}_1 - (I_1 - I_3)\,\omega_2 \omega_3 = 0 \\ I_1 \dot{\omega}_2 - (I_3 - I_1)\,\omega_3 \omega_1 = 0 \\ I_3 \dot{\omega}_3 = 0 \end{cases}$$

we assume $I_1 = I_2$

Here we assume that the body's center of mass is at rest and located at the origin of our coordinate system.

The trird equation above is decoupled from the other two and is easy to solve:

$$\omega_3(t) = \text{const}$$

The first and second equations can be written as

$$\dot{\omega}_1 = -\left(\frac{I_3 - I_1}{I_1}\,\omega_3\right)\omega_2$$

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\,\omega_3\right)\omega_1$$

If we define $\Omega = \dfrac{I_3 - I_1}{I_1}\,\omega_3$ they can be rewritten as

$$\begin{cases} \dot{\omega}_1 + \Omega\,\omega_2 = 0 \\ \dot{\omega}_2 - \Omega\,\omega_1 = 0 \end{cases}$$

The solution may be obtained by substituting

$$\omega_1 = \frac{\dot{\omega}_2}{\Omega} \Rightarrow \dot{\omega}_1 = \frac{\ddot{\omega}_2}{\Omega} \Rightarrow \ddot{\omega}_2 + \Omega \omega_2 = 0$$

Then

$$\omega_2(t) = A \sin(\Omega t + \alpha) \qquad (*)$$

$$\omega_1(t) = \frac{\dot{\omega}_2}{\Omega} = A \cos(\Omega t + \alpha)$$

where A and α are constants that could be determined from the initial conditions

$\omega_1(0)$ and $\omega_2(0)$

Note that

$$|\vec{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{A^2 + \omega_3^2} = \text{const}$$

Equations $(*)$ are parametric equations of a circle. The projection of $\vec{\omega}$ onto the xy plane describes a circle with time. The z-axis is the symmetry axis of the body. Vector $\vec{\omega}$ precesses about the z-axis with a constant angular frequency Ω. For an observer in the body frame $\vec{\omega}$ traces out a cone about the body symmetry axis.

Since in the above case we assumed a torque-free motion, $\vec{L} = \text{const}$ in the fixed coordinate system. There is another integral of motion in this system – the kinetic energy

$$T_{rot} = \frac{1}{2} \vec{L} \cdot \vec{\omega}$$

Since $\vec{L} = \text{const}$, $\vec{\omega}$ must precess in such a way that it projection on $\vec{L}$ is constant. Hence, $\vec{\omega}$ makes a constant angle with vector $\vec{L}$. Moreover we can show that $\vec{L}$ (in the fixed frame), $\vec{\omega}$ and z-axis (in the body frame) lie in one plane. This is because

$$\vec{L}\cdot(\vec{\omega}\times\hat{z}) = \vec{L}\cdot(\omega_2\hat{x}-\omega_1\hat{y}) = (I\vec{\omega})\cdot(\omega_2\hat{x}-\omega_1\hat{y})$$

$$= (I_1\omega_1\hat{x}+I_2\omega_2\hat{y}+I_3\omega_3\hat{z})\cdot(\omega_2\hat{x}-\omega_1\hat{y}) =$$

$$= (I_1-I_2)\omega_1\omega_2 = 0$$

The geometric interpretation is such that ω traces out a ~~body~~ cone and the body cone is rolling around the space cone.