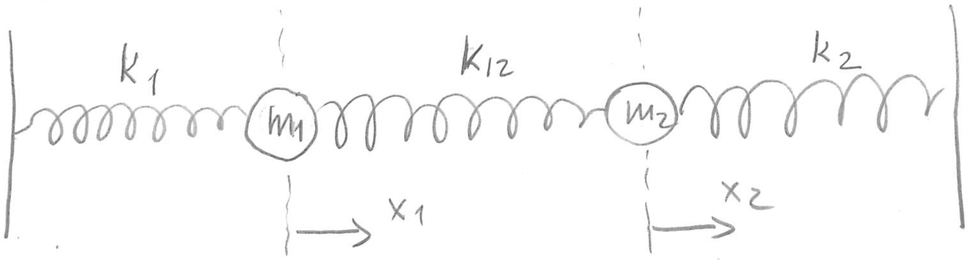
Two coupled harmonic oscillators

Before we dig into the general theory of small vibrations it is instructive to consider the case of (only) two coupled harmonic oscillators such as depicted in the figure below



Each mass is attached to a wall with a spring of force constant k_i The two masses are also connected with a spring of force constant k_{12} The motion is restricted to one dimension. Each coordinate is measured from the position of equilibrium.

If m_1 and m_2 are displaced by x_1 and x_2 respectively, the force on m_1 is $-k_1x_1 - k_{12}(x_1 - x_2)$ and the force on m_2 is $-k_2x_2 - k_{12}(x_2 - x_1)$. The equations of motions are then

$$\begin{cases} m_1\ddot{x}_1 + (k_1+k_{12})x_1 - k_{12}x_2 = 0 \\ m_2\ddot{x}_2 + (k_2+k_{12})x_2 - k_{12}x_1 = 0 \end{cases}$$

Since we have two linear differential equations with constant coefficient we can seek the solution in the form $Ae^{\lambda t}$. However, because we expect the solution to be oscillatory we can emphasise that

λ is imaginary right away and write instead

$$x_1(t) = a_1 e^{i\omega t} \qquad (*)$$
$$x_2(t) = a_2 e^{i\omega t}$$

We are to determine the frequencies ω and amplitudes a_1 and a_2. Note that a_1 and a_2 are complex numbers. So effectively each a_i contains two real constants that appear in the solution of a second order differential equation. Indeed

$$a e^{i\omega t} = \underbrace{\alpha}_{\text{magnitude}} \overset{\text{phase}}{e^{i\delta}} e^{i\omega t} = \alpha e^{i(\omega t+\delta)} =$$

$$= \alpha \cos(\omega t+\delta) + i\alpha \sin(\omega t+\delta)$$

We are certainly looking for real solutions ($x(t)$ is a trajectory after all). So instead of $A e^{i\omega t}$ we could seek the solutions in the form

$$x(t) = \alpha \cos(\omega t+\delta) \quad \text{or} \quad x(t) = \alpha \sin(\omega t+\delta)$$

However, it is more convenient to deal with exponents rather than with trigonometric functions and we keep the solutions $x_i(t)$ complex and at the end take the real (or imaginary part). Such a trick is possible thanks to the linearity of our differential equations. Changing the order of taking the real (imaginary) parts and multiplication by a constant or taking a time derivative has no effect on the final result.

Anyhow, after inserting our ansatz (*) into the equations of motion we get

$$-m_1\omega^2 a_1 e^{i\omega t} + (k_1+k_{12}) a_1 e^{i\omega t} - k_{12} a_2 e^{i\omega t} = 0$$

$$-m_2\omega^2 a_2 e^{i\omega t} + (k_2+k_{12}) a_2 e^{i\omega t} - k_{12} a_1 e^{i\omega t} = 0$$

Notice that these equations can be beautifully written in the matrix form. If we denote

$$x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{i\omega t} = \vec{a}\, e^{i\omega t}$$

then the equations of motion are

$$M\ddot{x} + Kx = 0 \qquad (**)$$

where

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad \text{— mass matrix}$$

$$K = \begin{pmatrix} k_1+k_{12} & -k_{12} \\ -k_{12} & k_2+k_{12} \end{pmatrix} \quad \text{— spring-constant matrix}$$

We can write (**) as

$$-\omega^2 M\vec{a}\, e^{i\omega t} + K\vec{a}\, e^{i\omega t} = 0$$

Cancelling the common exponential factor yields

$$(K-\omega^2 M)\vec{a} = 0 \quad \text{or} \quad K\vec{a} = \omega^2 M\vec{a}$$

This is nothing but a generalized eigenvalue problem with 2x2 matrices. K and M. ω^2 are the eigenvalues (ω_i^2's are presumably real) and

$\vec{a}$'s are eigenvectors. There are two solutions for a 2×2 problem.

If matrix $K-\omega^2 M$ has nonzero determinant then the only possible solution is a trivial one, $\vec{a}=0$, which corresponds to no motion at all. On the other hand, if

$$\det(K-\omega^2 M)=0$$

then non-trivial solutions exist.

Now let us consider a special case when

$m_1=m_2=m$ (masses are equal)

$k_1=k_2=k$ (all springs are the same)

this choice makes further analysis simpler.

$$\det(K-\omega^2 M) = \det\left[\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right] =$$

$$= \det\left[\begin{pmatrix} 2k-m\omega^2 & -k \\ -k & 2k-m\omega^2 \end{pmatrix}\right] = (2k-m\omega^2)^2 - k^2 = 0$$

or

$$2k - m\omega^2 = \pm k \quad \Rightarrow \quad \omega^2 = \frac{2k\pm k}{m} \qquad \omega_1=\sqrt{\frac{k}{m}}$$

$$\omega_2=\sqrt{\frac{3k}{m}}$$

ω_1 and ω_2 are the two normal frequencies at which our two masses can oscillate in purely sinusoidal manner. The sinusoidal motion with any one of the normal frequencies is called normal mode

Now let us find the eigenvectors of the generalized eigenvalue problem and see how exactly the system may oscillate

For $\omega_1 = \sqrt{\frac{k}{m}}$ $\quad K - \omega_1^2 M = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \begin{matrix} a_1 - a_2 = 0 \\ -a + a_2 = 0 \end{matrix} \quad \Rightarrow a_1 = a_2$$

$$x(t) = \begin{pmatrix} a \\ a \end{pmatrix} e^{i\omega_1 t} = \underset{\uparrow\ \text{real}}{\begin{pmatrix} \alpha \\ \alpha \end{pmatrix}} e^{i\delta} e^{i\omega_1 t}$$

$$Re[x(t)] = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \cos(\omega_1 t + \delta) \quad \text{or} \quad \begin{matrix} Re[x_1(t)] = \alpha \cos(\omega_1 t + \delta) \\ Re[x_2(t)] = \alpha \cos(\omega_1 t + \delta) \end{matrix}$$

We can see that in this first normal mode the two masses oscillate in phase with the same amplitude. This mode is symmetric

For $\omega_2 = \sqrt{\frac{3k}{m}}$ $\quad K - \omega_2^2 M = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \quad \Rightarrow \quad a_2 = -a_1$$

$$x(t) = \begin{pmatrix} b \\ -b \end{pmatrix} e^{i\omega_2 t} = \begin{pmatrix} \beta \\ -\beta \end{pmatrix} e^{i\lambda} e^{i\omega_2 t}$$

$$Re[x(t)] = \begin{pmatrix} \beta \\ -\beta \end{pmatrix} \cos(\omega_2 t + \lambda) \quad \text{or} \quad \begin{matrix} Re[x_1(t)] = \beta \cos(\omega_2 t - \lambda) \\ Re[x_2(t)] = -\beta \cos(\omega_2 t + \lambda) \end{matrix}$$

In the second normal mode the masses oscillate exactly out of phase. Hence this mode is antisymmetric

The general solution to the equations of motion is given by a linear combination

$$X(t) = a\begin{pmatrix}1\\1\end{pmatrix}e^{i\omega_1 t} + b\begin{pmatrix}1\\-1\end{pmatrix}e^{i\omega_2 t}$$

and

$$\mathrm{Re}\left[X(t)\right] = \alpha\begin{pmatrix}1\\1\end{pmatrix}\cos(\omega_1 t+\delta) + \beta\begin{pmatrix}1\\-1\end{pmatrix}\cos(\omega_2 t+\lambda)$$

The general solution, unlike the normal modes, are hard to describe or visualize is simple terms – it may look quite sophisticated even for this simple system of two equal masses.

General case of n coupled harmonic oscillators

Let us consider a case when we have n degrees of freedom. Suppose the generalized coordinates we deal with are

$$\vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

Let us assume that our system is conservative and its potential energy is

$$V(q_1, \dots, q_n) = V(\vec{q})$$

Given the relation between the Cartesian coordinates and the generalized coordinates

$$x_i = x_i(q_1 \dots q_n) \qquad \dot{x}_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \dot{q}_j$$

we can write

$$x_i^2 = \left(\sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \dot{q}_j \right) \left(\sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} \dot{q}_k \right)$$

The kinetic energy then takes the form

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{x}_i^2 = \frac{1}{2} \sum_{j,k=1}^{n} A_{jk}(\vec{q}) \dot{q}_j \dot{q}_k$$

where

$$A_{jk} = A_{jk}(q_1, \dots q_n) = \sum_{i=1}^{n} m_i \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_k}$$

Lastly, we will assume that the system is making small oscillations about its equilibrium configuration. If necessary, we can always redefine the coordinates such that the equilibrium is at $\vec{q} = 0$. The potential energy then can be Taylor-expanded around $\vec{q} = 0$:

$$V(\vec{q}) = V(\vec{0}) + \sum_{j=1}^{n} \frac{\partial V}{\partial q_j}\bigg|_{\vec{q}=0} q_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial V}{\partial q_j \partial q_k}\bigg|_{\vec{q}=0} q_j q_k + \dots$$

Now $V(\vec{0})$ can be dropped from further consideration as it is just a constant. All $\frac{\partial V}{\partial q_j}\Big|_{\vec{q}=0}$ terms vanish at equilibrium. We are left with

$$V = V(\vec{q}) \approx \frac{1}{2} \sum_{j,k=1}^{n} K_{jk} q_j q_k$$

We want to keep only terms up to the first non-vanishing order. This simplifies our kinetic energy. We can ignore everything but the zero-order terms in the Taylor expansion of $A_{jk}(\vec{q})$:

$$A_{jk}(\vec{q}) = M_{jk} + \underbrace{\sum_{l=1}^{n} \frac{\partial A_{jk}}{\partial q_l} q_l + \ldots}_{\text{ignore}}$$

Then

$$T = T(\dot{\vec{q}}) = \frac{1}{2} \sum_{j,k=1}^{n} M_{jk} \dot{q}_j \dot{q}_k$$

and the Lagrangian is

$$L(\vec{q}, \dot{\vec{q}}) = T(\dot{\vec{q}}) - V(\vec{q}) = \frac{1}{2} \sum_{j,k=1}^{n} M_{jk} \dot{q}_j \dot{q}_k - \frac{1}{2} \sum K_{jk} q_j q_k$$

The equations of motion corresponding to this Lagrangian are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad i = 1, \ldots, n$$

$$\frac{\partial L}{\partial q_i} = \sum_{j=1}^{n} K_{ij} q_j \qquad \frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{n} M_{ij} \dot{q}_j \qquad i = 1, \ldots, n$$

so

$$\sum_{j=1}^{n} M_{ij} \ddot{q}_j + \sum_{j=1}^{n} K_{ij} q_j = 0 \qquad i = 1, \ldots, n$$

In the matrix form it looks as follows:

$$M\ddot{\vec{q}} + K\vec{q} = 0 \qquad \vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

where M and K are $n \times n$ matrices. We can seek the solution of the above equation in the form

$$\vec{q} = \mathrm{Re}\left[\vec{a}\, e^{i\omega t}\right] \quad \text{where} \quad \vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{is a constant vector-column}$$

That leads to the following generalized eigenvalue problem:

$$\left(K - \omega^2 M\right)\vec{a} = 0$$

which has nontrivial solutions only if ω satisfies the secular equation

$$\det\left[K - \omega^2 M\right] = 0$$

The determinant yields an n-th order polynomial in ω^2. Hence get n solutions, which are the normal frequencies of the system. The corresponding eigenvectors define the normal modes