Energy of a continuous vibrating string

Let us considerr a vibrating string. If there is no dissipation in the system the total mechanical energy must remain constant. From the previous lecture we know that the solution of the string equation of motion can be represented as

$$q(x,t) = \sum_{s=1}^{\infty}{}' \left[\beta_s \cos\omega_s t + \gamma_s \sin\omega_s t\right] \sin\left(\frac{s\pi x}{L}\right)$$

where $\omega_s = \frac{s\pi}{L}\sqrt{\frac{\tau}{\rho}}$ $\quad s = 1, 2, 3, \ldots$ $\quad$ τ - tension, ρ - linear density

We can also write it simply as

$$q(x,t) = \sum_{s=1}^{\infty}{}' \eta_s(t) \sin\left(\frac{s\pi x}{L}\right) \qquad \eta_s(t) = \mathrm{Re}\left[a_s e^{i\omega_s t}\right] \quad (*)$$

The kinetic energy of an element of the sting dm is $\frac{1}{2} dm \cdot \dot{q}^2 = \frac{1}{2}\rho dx\, \dot{q}^2$. Integrated over the entire length it gives the total kinetic energy:

$$T = \frac{1}{2}\rho \int_0^L \left(\frac{\partial q}{\partial t}\right)^2 dx$$

where we assumed that the linear density ρ is constant.

That is

$$T = \frac{1}{2}\rho \int_0^L \left[\frac{d}{dt}\sum_s{}' \eta_s \sin\left(\frac{s\pi x}{L}\right)\right]^2 dx =$$

$$= \frac{1}{2}\rho \sum_{r,s} \dot{\eta}_r \dot{\eta}_s \underbrace{\int_0^L \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{s\pi x}{L}\right) dx}_{\frac{L}{2}\delta_{rs}} = \frac{\rho L}{4}\sum_{r,s} \dot{\eta}_r \dot{\eta}_s \delta_{rs}$$

$$= \frac{\rho L}{4}\sum_s{}' \dot{\eta}_s^2$$

Now

$$\dot{\eta}_s^2 = \left[\frac{d}{dt}\left(\beta_s \cos\omega_s t + \gamma_s \sin\omega_s t\right)\right]^2 =$$

$$= \left[-\beta_s\omega_s \sin\omega_s t + \gamma_s\omega_s \cos\omega_s t\right]^2 = \omega_s^2\left(\beta_s \sin\omega_s t - \gamma_s \cos\omega_s t\right)^2$$

Therefore,

$$T = \frac{\rho L}{4}\sum_s \omega_s^2\left(\beta_s \cos\omega_s t - \gamma_s \sin\omega_s t\right)^2$$

The potential energy of the linear array of coupled oscillators is given by (see previous lecture):

$$V = \frac{1}{2}\frac{\tau}{b}\left[q_1^2 + (q_2-q_1)^2 + \dots + (q_n - q_{n-1})^2 + q_n^2\right]$$

In the limit $b = dx \to 0$, $n \to \infty$ it becomes.

$$V = \frac{1}{2}\tau \sum_j \frac{(q_j - q_{j-1})^2}{dx^2}\,dx = \frac{1}{2}\tau\int_0^L \left(\frac{\partial q}{\partial x}\right)^2 dx$$

Inserting expression (*) in place of q then yields for $\frac{\partial q}{\partial x}$:

$$\frac{\partial q}{\partial x} = \sum_s \eta_s \frac{s\pi}{L}\cos\frac{s\pi x}{L}$$

and for V:

$$V = \frac{\tau}{2}\int_0^L\left[\sum_s \eta_s\frac{s\pi}{L}\cos\frac{s\pi x}{L}\right]^2 dx = \frac{\tau}{2}\sum_{r,s}\eta_r\eta_s\frac{r\pi}{L}\frac{s\pi}{L}\underbrace{\int_0^L \cos\frac{r\pi x}{L}\cos\frac{s\pi x}{L}\,dx}_{\frac{L}{2}\delta_{rs}}$$

$$= \frac{\tau}{2}\sum_{r,s}\eta_s\eta_r\frac{r\pi}{L}\frac{s\pi}{L}\frac{L}{2}\delta_{rs} = \frac{\tau L}{4}\sum_s\frac{s^2\pi^2}{L^2}\eta_s^2 =$$

$$= \frac{\rho L}{4}\sum_s \omega_s^2\eta_s^2$$

With that we can write

$$V = \frac{\rho L}{4} \sum_s \omega_s^2 (\beta_s \cos\omega_s t + \gamma_s \sin\omega_s t)^2$$

The total mechanical energy is the sum of T and V:

$$E = \frac{\rho L}{4} \sum_s \omega_s^2 \left[(\beta_s \sin\omega_s t - \gamma_s \cos\omega_s t)^2 + (\beta_s \cos\omega_s t + \gamma_s \sin\omega_s t)^2 \right] =$$

$$= \frac{\rho L}{4} \sum_s \omega_s^2 (\beta_s^2 + \gamma_s^2) \quad \leftarrow$$ time-independent expression

$$= \frac{\rho L}{4} \sum_s \omega_s^2 |a_s|^2 \quad \leftarrow$$ each normal mode gives its own separate contribution

Let us now compute time-averaged kinetic energy and time-averaged potential energy:

$$\overline{T} = \frac{1}{t_f} \int_0^{t_f} T \, dt \qquad \overline{V} = \frac{1}{t_f} \int_0^{t_f} V \, dt$$

Given that the time averages of the following functions are:

$$\overline{\sin^2 \omega_s t} = \frac{1}{2} \qquad \overline{\cos^2 \omega_s t} = \frac{1}{2} \qquad \overline{\sin\omega_s t \cos\omega_s t} = 0$$

we get

$$\overline{T} = \frac{\rho L}{8} \sum_s \omega_s^2 (\beta_s^2 + \gamma_s^2)$$

$$\overline{V} = \frac{\rho L}{8} \sum_s \omega_s^2 (\beta_s^2 + \gamma_s^2)$$

Thus

$$\overline{T} = \overline{V} = \frac{1}{2} E$$

which is a manifestation of the virial theorem for a system of harmonic oscillators.

Wave equation

Let us take the equations of motion for a linear array of coupled oscillators and take their limit when $n \to \infty$ and $b \to 0$. In the previous lecture we had

$$m\ddot{q}_j = -\frac{\tilde{\tau}}{b}(q_j - q_{j-1}) + \frac{\tilde{\tau}}{b}(q_{j+1} - q_j) \qquad j = 2, \dots, n-1$$

If we divide both sides by b then

$$\underbrace{\frac{m}{b}}_{\rho}\ddot{q}_j = \frac{\tilde{\tau}}{b}\left[\underbrace{\frac{q_{j+1} - q_j}{b}}_{\left.\frac{\partial q}{\partial x}\right|_{x_j + \frac{b}{2}}} - \underbrace{\frac{q_j - q_{j-1}}{b}}_{\left.\frac{\partial q}{\partial x}\right|_{x_j - \frac{b}{2}}}\right] = \tau \left.\frac{\partial^2 q}{\partial x^2}\right|_{x_j}$$

Therefore

$$\rho\ddot{q} = \tau\frac{\partial^2 q}{\partial x^2} \quad \text{or simply}$$

$$\frac{\partial^2 q}{\partial x^2} = \frac{\rho}{\tau}\frac{\partial^2 q}{\partial t^2} \longleftarrow \text{the wave equation}$$

The coefficient $\frac{\rho}{\tau}$ has the units of inverse vecolity. Hence, we can formally define $\frac{\rho}{\tau} = \frac{1}{v^2}$ or $v = \sqrt{\frac{\tau}{\rho}}$

To show that the above equation does indeed desribe the motion of waves let us make a substitution:

$$\xi \equiv x + vt \qquad \eta \equiv x - vt$$

with this substitutions the derivatives with respect to x and t become

$$\frac{\partial q}{\partial x} = \frac{\partial q}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial q}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial q}{\partial \xi} + \frac{\partial q}{\partial \eta}$$

$$\frac{\partial^2 q}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial q}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial q}{\partial \xi} + \frac{\partial q}{\partial \eta}\right) = \frac{\partial^2 q}{\partial \xi^2} + 2\frac{\partial^2 q}{\partial \xi \partial \eta} + \frac{\partial^2 q}{\partial \eta^2}$$

$$\frac{\partial q}{\partial t} = \frac{\partial q}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial q}{\partial \eta}\frac{\partial \eta}{\partial t} = v\frac{\partial q}{\partial \xi} - v\frac{\partial q}{\partial \eta}$$

$$\frac{\partial^2 q}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial q}{\partial t}\right) = v\frac{\partial}{\partial t}\left(\frac{\partial q}{\partial \xi} - \frac{\partial q}{\partial \eta}\right) = v^2\left(\frac{\partial^2 q}{\partial \xi^2} - 2\frac{\partial^2 q}{\partial \xi \partial \eta} + \frac{\partial^2 q}{\partial \eta^2}\right)$$

Looking at the wave equation we see that

$$\frac{\partial^2 q}{\partial \xi^2} + 2\frac{\partial^2 q}{\partial \xi d\eta} + \frac{\partial^2 q}{\partial \eta^2} = \frac{\partial^2 q}{\partial \xi^2} - 2\frac{\partial^2 q}{\partial \xi d\eta} + \frac{\partial^2 q}{\partial \eta^2}$$

which gives us

$$\frac{\partial^2 q}{\partial \xi d\eta} = 0$$

The most general solution of this last equation is

$$q = f(\xi) + g(\eta)$$

where f and g are some arbitrary functions

Thus,

$$q = f(x+vt) + g(x-vt)$$

Function $f(x+vt)$ represents some form/shape that propagates along the negative direction of the x-axis, while function $g(x-vt)$ represents some form/shape that propagates along the positive direction of the x-axis

Method of separation of variables (the Fourier method)

Let us consider a technique of solving ordinary and partial differential equations commonly used in mathematical physics that is called the method of separation of variables. We will apply this technique to our wave equation.

$$\frac{\partial^2 q}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 q}{\partial t^2} \qquad (**)$$

subject to the boundary $q(0)=q(L)=0$ and initial

$$q(x,t=0) = q(x,0) = \phi(x) \qquad 0 \le x \le L$$

$$\dot{q}(x,t=0) = \dot{q}(x,0) = \psi(x) \qquad 0 \le x \le L$$

conditions.

The idea of the approach is to find <u>separated</u> solutions in the form $q(x,t) = X(x)\Theta(t)$. Of course there is absolutely no reason to think that the general solution of equation (**) is of this separated form. However, if we find a multitude of separate solutions we can use the linearity of (*), which results in the superposition principle: a sum of several solutions is a solution. We can then try to represent the general solution as a series in terms of separated solutions.

If we substitute $q(x,t) = X(x)\Theta(t)$ into (*) we obtain

$$X''(x)\,\Theta(t) = \frac{1}{v^2} X(x)\,\ddot{\Theta}(t)$$

or

$$\frac{X''}{X} = \frac{1}{v^2}\frac{\ddot{\Theta}}{\Theta}$$

The latter relation must hold for any value of x or t. That is possible only if $\frac{X''}{X} = -\frac{1}{v^2}\omega^2$ and $\frac{\ddot{\Theta}}{\Theta} = -\omega^2$, where

ω^2 is a constant (called the separation constant)

For $\Theta(t)$ we then have the equation

$$\ddot{\Theta} + \omega^2 \Theta = 0$$

which can be solved easily: $\Theta(t) = Ce^{i\omega t} + De^{-i\omega t}$

For $X(x)$ we have the eigenvalue problem

$$X'' + \frac{\omega^2}{v^2} X = 0 \qquad X(0) = X(L) = 0$$

The solution of $X'' + \frac{\omega^2}{v^2} X = 0$ is:

$$X(x) = F \sin \frac{\omega}{v} x + G \cos \frac{\omega}{v} x$$

The boundary condition $X(0) = 0$ gives $G = 0$, while the other boundary condition, $X(L) = 0$, yields:

$$\sin\left(\frac{\omega}{v} L\right) = 0 \quad \Rightarrow \quad \omega_n = \left(\frac{n\pi}{L}\right) v \qquad n = \pm 1, \pm 2, \pm 3, \dots$$

If we define k (the wave number) as

$$k^2 = \frac{\omega^2}{v^2} = \frac{n^2 \pi^2}{L^2}$$

we can write the separated solutions as

$$q_n(x,t) = e^{\pm i k_n x} \cdot e^{\pm i k_n v t} = e^{\pm i k_n (x \pm v t)}$$

or

$$q_n(x,t) = \sin k_n x \, e^{\pm i \omega_n t}$$

The general solution of the wave equation can then be sought as

$$q(x,t) = \sum_n \left(A_n e^{i\omega_n t} + B_n e^{-i\omega_n t} \right) \sin k_n x$$

Coefficients A_n and B_n can be determined from the initial conditions

$$q(x,0) = \phi(x) = \sum_n{}' (A_n + B_n) \sin k_n x$$

$$\int_0^L \phi(x) \sin k_m x \, dx = \sum_n{}' (A_n + B_n) \underbrace{\int_0^L \sin k_n x \sin k_m x \, dx}_{\frac{L}{2}\delta_{nm}}$$

so

$$(A_m + B_m) = \frac{2}{L} \int_0^L \phi(x) \sin k_m x \, dx$$

while

$$\dot{q}(x,0) = \psi(x) = \sum_n{}' i\omega_n (A_n - B_n) \sin k_n x$$

$$\int_0^L \psi(x) \sin k_m x \, dx = i \sum_n \omega_n (A_n - B_n) \underbrace{\int_0^L \sin k_n x \sin k_m x \, dx}_{\frac{L}{2}\delta_{nm}}$$

and

$$(A_m - B_m) = \frac{1}{i\omega_m} \frac{2}{L} \int_0^L \psi(x) \sin k_m x \, dx$$

Solving for A_m and B_m gives:

$$A_m = \frac{1}{L} \left(\int_0^L \phi(x) \sin k_m x \, dx + \frac{1}{i\omega_m} \int_0^L \psi(x) \sin k_m x \, dx \right)$$

$$B_m = \frac{1}{L} \left(\int_0^L \phi(x) \sin k_m x \, dx - \frac{1}{i\omega_m} \int_0^L \psi(x) \sin k_m x \, dx \right)$$