Group velocity and wave packets

Let us use the superposition principle and add two particular solutions to the wave equation, ψ_1 and ψ_2, that in some sense very close to each other:

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\psi_1(x,t) = A\, e^{i(\omega_1 t - k_1 x)}$$

$$\psi_2(x,t) = A\, e^{i(\omega_2 t - k_2 x)}$$

Here we will assume that

$$\omega_2 = \omega_1 + \Delta\omega \qquad \omega_1 \equiv \omega$$

$$k_2 = k_1 + \Delta k \qquad k_1 \equiv k$$

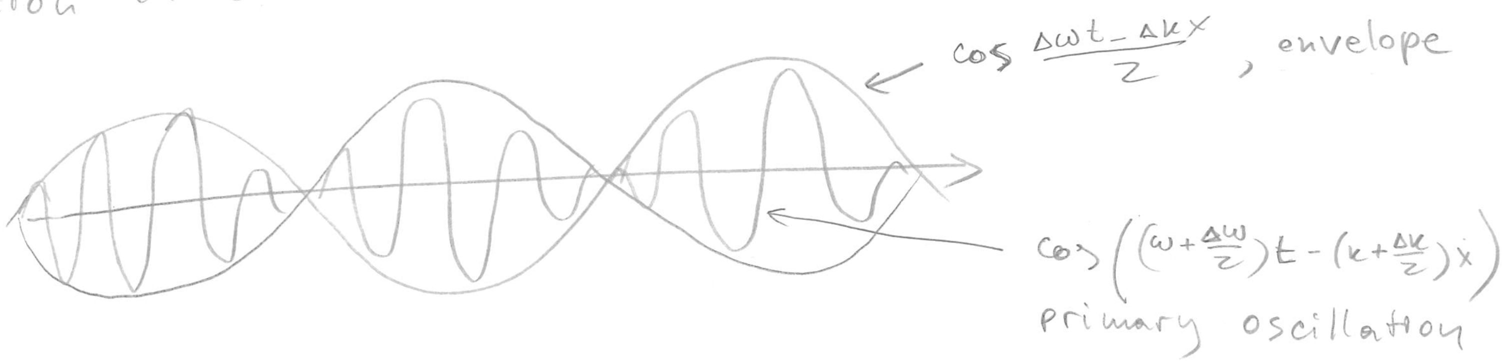
The sum of the two solutions, which is also a solution, is

$$\psi(x,t) = \psi_1 + \psi_2 = A\left[e^{i\omega t} e^{-ikx} + e^{i(\omega+\Delta\omega)t} e^{-i(k+\Delta k)x}\right] =$$

$$= A e^{i(\omega + \frac{\Delta\omega}{2})t} e^{-i(k+\frac{\Delta k}{2})x}\left[e^{-i\left(\frac{\Delta\omega t}{2} - \frac{\Delta k x}{2}\right)} + e^{i\left(\frac{\Delta\omega t}{2} - \frac{\Delta k x}{2}\right)}\right] =$$

$$= 2A e^{i(\omega+\frac{\Delta\omega}{2})t} e^{-i(k+\frac{\Delta k}{2})x} \cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)$$

Since in practice we take the real part of the solution in the end, we get the phenomenon of beating



$$\mathrm{Re}\left[\psi(x,t)\right] = 2A\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)\cos\left(\left(\omega+\frac{\Delta\omega}{2}\right)t - \left(k+\frac{\Delta k}{2}\right)x\right)$$

The velocity with which the envelope propagates is called the group velocity v_g. It is given by the requirement that the phase of the amplitude/envelope term be constant

$$v_g = \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}$$

In a nondispersive medium $v_g = \frac{\Delta\omega}{\Delta k} = v$, $v_p = \frac{\omega}{k} = v$. So the group and phase velocities are the same. However, if dispersion is present, i.e. if $v_g = v_g(k)$ and $v_p = v_p(k)$ then they are generally distinct.

Now let us generalize things and consider the superposition of n waves, not just two:

$$\psi(x,t) = \sum_{s=1}^{n} A_s e^{i(\omega_s t - k_s x)}$$

Amplitudes A_s may not be the same now. When n becomes very large ($n \to \infty$) and the string is infinite ($L \to \infty$) the spacing between eigenfrequencies becomes infinitesimal and they become continuously distributed. So we replace the sum with an integral

$$\psi(x,t) = \int_{-\infty}^{+\infty} A(k) e^{i(\omega t - kx)} dk$$

$A(k)$ represents the spectral distribution of the plane waves included in our $\psi(x,t)$. Of particular importance is the case when function $A(k)$ is localized in the neighborhood of some value k_0, while being vanishingly small outside a small interval $[k_0 - \Delta k, k_0 + \Delta k]$. Then we can write

$$\psi(x,t) = \int_{k_0-\Delta k}^{k_0+\Delta k} A(k) e^{i(\omega t - kx)} dk$$

Such $\psi(x,t)$ is called a wave packet. The concept of group velocity can only be applied meaningfully to those cases when $\psi(x,t)$ is a wave packet, i.e. when $A(k)$ is nonvanishing in a small range of k values. In the case of a wave packet we can represent $\omega(k)$ as a Taylor series:

$$\omega(k) = \omega(k_0) + \left.\frac{d\omega}{dk}\right|_{k=k_0} \cdot (k-k_0) + \frac{1}{2}\left.\frac{d^2\omega}{dk^2}\right|_{k=k_0} (k-k_0)^2 + \dots$$

and limit ourselves with the first two terms only, i.e.

$$\omega(k) \approx \omega_0 + \omega_0'(k-k_0) \qquad \omega_0 \equiv \omega(k_0)$$

Then in the exponent inside the integral expression for the wave packet the argument is

$$\omega t - kx \simeq \omega_0 t + \omega_0'(k-k_0)t - kx - k_0x + k_0x =$$

$$= (\omega_0 t - k_0 x) + \omega_0'(k-k_0)t - (k-k_0)x =$$

$$= (\omega_0 t - k_0 x) + (k-k_0)(\omega_0' t - x)$$

So that

$$\psi(x,t) = \underbrace{\int_{k_0-\Delta k}^{k_0+\Delta k} A(k)\, e^{i(k-k_0)(\omega_0' t - x)}}_{\substack{\text{effective amplitude}\\ \text{that varies very slowly}}} e^{i(\omega_0 t - k_0 x)}\, dk$$

$A(k)\, e^{i(k-k_0)(\omega_0' t - x)}$ is similar to $2A\cos\left(\frac{\Delta\omega t - \Delta k x}{2}\right)$ term in the case when we added just two solutions. The requirement of constant phase ($d\phi = 0$) for the amplitude term yields the group velocity

$$v_g = \omega_0' = \left.\left(\frac{d\omega}{dk}\right)\right|_{k=k_0}$$