Relativistic momentum and energy

Newton's second law $\vec{F} = m \frac{d^2 \vec{r}}{dt^2}$ is covariant under Galilean transformation. That is it looks the same in two different inertial reference frames. Inded, consider reference frame K moving with relative velocity v relative to frame K'. If a particle of mass m undergoes a constant acceleration in K' in the x-direction, then

$$F' = m'a' = m' \frac{d^2 x'}{dt'^2} = m' \frac{d}{dt'} \left(\frac{dx'}{dt'} \right)$$

Given that

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t \qquad m' = m$$

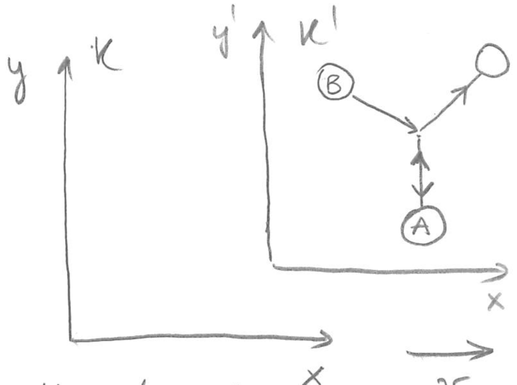
we have

$$F' = m \frac{d}{dt} \left(\frac{d(x - vt)}{dt} \right) = m \frac{d}{dt} (u - v) = m \frac{du}{dt} = ma = F$$

So no matter which inertial reference frame we observe from, Newton's second law looks the same.

We do not expect that the same Newton's second law remains covariant under the Lorentz transform. Therefore we cannot use it to find trajectories. Neither we can use the law of conservation of energy, unless we make some changes.

Let us consider the conservation of momentum in a collision. Observer A holds a ball of mass m in frame K, while observer B holds another ball of mass m in frame K'. The relative velocity of K' with respect to K is v. The two observers throw their balls along their y-axes. It results in a perfectly elastic collision. Each observer measures the speed of his/her ball to be u_0 (in opposite direction in K and K')



In frame K we have for ball A

$$u_{Ax} = 0$$

$$u_{Ay} = u_0$$

So the initial momentum of ball A is $p_{Ay} = m u_0$ and $p_{Ax} = 0$. As the collision is elastic the ball returns down with speed u_0. The momentum change is

$$\Delta p_{Ay} = -2 m u_0$$

Let us now apply the velocity transformation formula to determine the velocity of ball B in K

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \qquad u'_z = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

So in our case we interchange primes and unprimes and let $v \to -v$

$$u_{Bx} = \frac{0 + v}{1 + \frac{0 \cdot v}{c^2}} = v$$

$$u_{By} = -\frac{u_0}{\left(1 - \frac{0 \cdot v}{c^2}\right)\gamma} = -u_0 \sqrt{1 - \frac{v^2}{c^2}}$$

(we used $u'_{Bx} = 0$ and $u'_{By} = -u_0$)

Then we can say that the momentum of ball B and its change are

$$p_{By} = -m u_0 \sqrt{1 - \frac{v^2}{c^2}}$$

$$\Delta p_{By} = +2 m u_0 \sqrt{1 - \frac{v^2}{c^2}}$$

As we can see $\Delta p_{Ay} \neq \Delta p_{By}$. Let us find a modification that will allow us to keep the momentum conservation and Newton's second law. Let us assume the simplest possible change and allow the following form

$$\vec{p} = k(u)\, m \vec{u}$$

With such a modification we can show that the momentum in y-direction is conserved if we use

$k(u) = 1/\sqrt{1 - u^2/c^2}$

Indeed $p_{Ay} = mu_0$ becomes

$$p_{Ay} = \frac{mu_0}{\sqrt{1-\frac{u_0^2}{c^2}}} \quad \text{and} \quad \Delta p_{Ay} = -\frac{2mu_0}{\sqrt{1-\frac{u_0^2}{c^2}}}$$

The magnitude of u_B vector as measured in K is

$$u_B = \sqrt{u_{Bx}^2 + u_{By}^2} = \sqrt{v^2 + u_0^2\left(1-\frac{v^2}{c^2}\right)}$$

Then the momentum p_{By} is (again we assume $\vec{p} = k(u)m\vec{u}$)

$$p_{By} = -mu_0\sqrt{1-\frac{v^2}{c^2}} \cdot \frac{1}{\sqrt{1-\frac{u_B^2}{c^2}}} = -mu_0\sqrt{\frac{1-\frac{v^2}{c^2}}{1-\frac{u_B^2}{c^2}}}$$

Substituting the expression for u_B then yields

$$p_{By} = \frac{-mu_0\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}-\frac{u_0^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}} = \frac{-mu_0\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u_0^2}{c^2}\right)}} = -\frac{mu_0}{\sqrt{1-\frac{u_0^2}{c^2}}}$$

and

$$\Delta p_{By} = \frac{+2mu_0}{\sqrt{1-\frac{u_0^2}{c^2}}}$$

so indeed we end up with $\Delta p_{Ay} = \Delta p_{By}$

We can compute the relativistic momentum if we use the proper time τ insted of t

$$\vec{p} = m\frac{d\vec{r}}{d\tau} = m\frac{d\vec{r}}{dt}\frac{dt}{d\tau}$$

recall that $t = \gamma\tau$ and for our case $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

Then

$$\vec{p} = m\frac{d\vec{r}}{dt}\frac{d}{d\tau}\frac{\tau}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}} \quad \leftarrow \text{definition of relativistic momentum}$$

Here we retain that $\vec{u} \equiv \frac{d\vec{r}}{dt}$

In the limit $u \ll c$ $\vec{p} = m\vec{u}$ as in the Newtonian mechanics.

Now let us turn our attention to the relativistic energy and force. If we stick to the definition of kinetic energy as the work done on a particle then

$$W = \int_1^2 \vec{F} \cdot d\vec{r} = T_2 - T_1$$

here we use $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}\right)$. If we start at rest $(T_1 = 0)$ and $\vec{u} \parallel \vec{F}$ we have

$$W = T = \int_1^2 \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right) \cdot \vec{u}\, dt = m\int_0^u u\, d\left(\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}\right) \overset{\text{by parts}}{=}$$

$$= \frac{mu^2}{\sqrt{1-\frac{u^2}{c^2}}} - m\int_0^u \frac{u\,du}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{mu^2}{\sqrt{1-\frac{u^2}{c^2}}} + mc^2\sqrt{1-\frac{u^2}{c^2}}\Bigg|_0^u =$$

$$= \frac{mu^2}{\sqrt{1-\frac{u^2}{c^2}}} + mc^2\sqrt{1-\frac{u^2}{c^2}} - mc^2 = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}} - mc^2$$

Therefore we define the relativistic kinetic energy as

$$T = \gamma mc^2 - mc^2 \qquad \text{where} \qquad \gamma \equiv \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

For $u \ll c$ we recover the result of the Newtonian mechanics

$$T = mc^2\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right) \approx \frac{1}{2}mu^2$$

The term mc^2 in the expression for T is called the rest energy, $E_0 \equiv mc^2$. If so then

$$T = \gamma mc^2 - E_0$$

and then we can define the total relativistic energy as

$$E = T + E_0 = \gamma mc^2$$

It is also possible to relate the total relativistic energy and relativistic momentum :

$$p \equiv \gamma m u \implies p^2c^2 = \gamma^2 m^2 u^2 c^2 = \gamma^2 m^2 c^4 \frac{u^2}{c^2} =$$

$$= \gamma^2 m^2 c^4 \underbrace{\left(1 - \frac{1}{\gamma^2}\right)}_{\frac{u^2}{c^2}} = \gamma^2 m^2 c^4 - m^2 c^4 = E^2 - E_0^2$$

This yields the very well-known relation

$$E^2 = p^2c^2 + E_0^2 = p^2c^2 + m^2c^4$$