1. The energy levels of infinite square wells of width a and 2a are:

$$E_n^{(a)} = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad E_n^{(2a)} = \frac{n^2\pi^2\hbar^2}{2m(2a)^2} = \frac{\left(\frac{n}{2}\right)^2\pi^2\hbar^2}{2ma^2}$$

It is easy to see that the ground state energy (n=1) in the well of width a is the same as the first excited (n=2) state energy in the well of width 2a. Therefore we need to determine the "contribution" of the first excited state $\psi_2^{(2a)}$ in the expansion of $\psi_1^{(a)}$ in terms of $\psi_k^{(2a)}$:

$$\psi_1^a(x) = \sum_{k=1}^{\infty} c_k \psi_k^{(2a)}(x) \qquad c_2 = \int_0^{2a} \psi_2^{(2a)}(x)\,\psi_1^{(a)}(x)\,dx$$

where

$$\psi_1^{(a)}(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\frac{\pi x}{a} & 0<x<a \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_2^{(2a)}(x) = \begin{cases} \sqrt{\frac{2}{2a}}\sin\frac{2\pi x}{2a} = \sqrt{\frac{1}{a}}\sin\frac{\pi x}{a} & 0<x<2a \\ 0 & \text{otherwise} \end{cases}$$

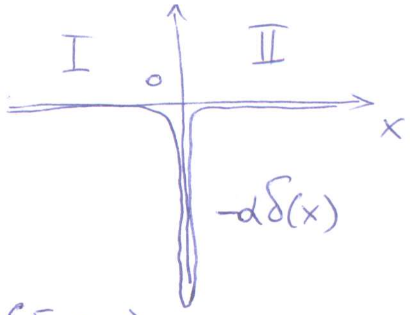
$$c_2 = \int_0^a \underbrace{\frac{1}{\sqrt{2}}\psi_1^{(a)}(x)}_{\psi_2^{(2a)}(x)}\,\psi_1^{(a)}\,dx = \frac{1}{\sqrt{2}}$$

The probability is equal to the absolute square of the expansion coefficient:

$$P = |c_2|^2 = \frac{1}{2}$$

2. The Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi$$



in region I $(x<0)$ becomes

$$\frac{d^2\psi_I}{dx^2} + k^2\psi_I = 0 \qquad \text{where } k \equiv \frac{\sqrt{-2mE}}{\hbar} \quad (E<0)$$

and the general solution of it is:

$$\psi_I(x) = Ae^{kx} + Be^{-kx}$$

Since the wave function must be square integrable we have to require $B = 0$.

Similarly, for region II $(x>0)$

$$\frac{d^2\psi_{II}}{dx^2} + k^2\psi_{II} = 0 \qquad k = \frac{\sqrt{-2mE}}{\hbar}$$

and

$$\psi_{II}(x) = Ce^{kx} + De^{-kx}$$

The square integrability again requires that $C = 0$

$\psi_I(x)$ and $\psi_{II}(x)$ can be written as one:

$$\psi(x) = \begin{cases} \psi_I(x) & x<0 \\ \psi_{II}(x) & x>0 \end{cases} = Ae^{-k|x|}$$

As the wave function must be normalized, i.e. $\int_{-\infty}^{+\infty} A^2 e^{-2k|x|}\,dx = 1$

we find that $A^2\frac{1}{k} = 1$ and $A = \frac{1}{\sqrt{k}}$

The continuity of the wave function requires that $\psi_I(0) = \psi_{II}(0)$.

We already have that satisfied. To determine the magnitude of the jump of the first derivative we can integrate the Schrödinger equation over an infinitely small region around the point $x=0$:

$$\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}\underbrace{\int_{-\epsilon}^{+\epsilon}V(x)\,dx}_{-\alpha} = -\frac{2m}{\hbar^2}E\underbrace{\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx}_{0}$$

$$\psi'(+\epsilon) - \psi'(-\epsilon) + \frac{2m\alpha}{\hbar^2} = 0$$

$$-\frac{1}{\sqrt{k}}k - \frac{1}{\sqrt{k}}k + \frac{2m\alpha}{\hbar^2} = 0 \implies \sqrt{k} = \frac{m\alpha}{\hbar^2} \implies -\frac{2mE}{\hbar^2} = \frac{m^2\alpha^2}{\hbar^4}$$

$$E = -\frac{m\alpha^2}{2\hbar^2}$$ $\leftarrow$ this is the only solution for negative energy values (bound states)

3. $\Psi(x,0) = A\left[\phi_0(x) + i\,\phi_1(x)\right]$

a) $1 = \int |\Psi(x,0)|^2 dx = |A|^2 \left(\underbrace{\int |\phi_0|^2 dx}_{1} + i \underbrace{\int \phi_0^* \phi_1 dx}_{0} - i \underbrace{\int \phi_1^* \phi_0 dx}_{0} + \underbrace{\int |\phi_1|^2 dx}_{1} \right)$

$A = \frac{1}{\sqrt{2}}$ (disregarding an arbitrary phase factor)

b) $\Psi(x,t) = \frac{1}{\sqrt{2}} \phi_0(x) e^{-\frac{iE_0 t}{\hbar}} + \frac{i}{\sqrt{2}} \phi_1(x) e^{-\frac{iE_1 t}{\hbar}}$

where $E_0 = \frac{\hbar\omega}{2}$ $E_1 = \frac{3}{2}\hbar\omega$ and ω is the angular frequency of the oscillator

c) $E_1 - E_0 = \hbar\omega$

For any hermitian operator, $\hat{O}$

$\langle \hat{O} \rangle = \langle \Psi(x,t) | \hat{O} | \Psi(x,t) \rangle = \frac{1}{2}\left[\langle \phi_0 | \hat{O} | \phi_0 \rangle + i e^{-i\omega t} \langle \phi_0 | \hat{O} | \phi_1 \rangle - i e^{i\omega t} \langle \phi_1 | \hat{O} | \phi_0 \rangle + \langle \phi_1 | \hat{O} | \phi_1 \rangle \right] = \frac{1}{2}\left[\langle \phi_0 | \hat{O} | \phi_0 \rangle + \langle \phi_1 | \hat{O} | \phi_1 \rangle + 2\,\mathrm{Re}\left[i e^{-i\omega t} \langle \phi_0 | \hat{O} | \phi_1 \rangle \right] \right] =$

$= \frac{1}{2}\left[\langle \phi_0 | \hat{O} | \phi_0 \rangle + \langle \phi_1 | \hat{O} | \phi_1 \rangle + 2\,\mathrm{Re}\left[(i\cos\omega t + \sin\omega t) \langle \phi_0 | \hat{O} | \phi_1 \rangle \right] \right]$

$\langle x \rangle = \frac{1}{2}\left[0 + 0 + 2\sin\omega t \sqrt{\frac{\hbar}{2m\omega}} \right] = \sqrt{\frac{\hbar}{2m\omega}} \sin\omega t$

$\langle p \rangle = \frac{1}{2}\left[0 + 0 - 2\cos\omega t \sqrt{\frac{m\hbar\omega}{2}} \right] = \sqrt{\frac{m\hbar\omega}{2}} \cos\omega t$

For $\hat{H}$ only the diagonal elements are non-zeros:

$\langle \phi_k | \hat{H} | \phi_n \rangle = E_n \delta_{kn}$

So

$\langle H \rangle = \frac{1}{2}\left[E_1 + E_2 \right] = \hbar\omega$

4. a) $\hat{P}|\psi\rangle = |\phi\rangle\langle\phi|\psi\rangle$

For linearity we need to show that $\hat{P}(\lambda_1\psi_1 + \lambda_2\psi_2) = \lambda_1\hat{P}\psi_1 + \lambda_2\hat{P}\psi_2$

$$\hat{P}|\lambda_1\psi_1 + \lambda_2\psi_2\rangle = |\phi\rangle\langle\phi|\lambda_1\psi_1 + \lambda_2\psi_2\rangle = \lambda_1 |\phi\rangle\langle\phi|\psi_1\rangle + \lambda_2|\phi\rangle\langle\phi|\psi_2\rangle =$$

$$= \lambda_1\hat{P}|\psi_1\rangle + \lambda_2\hat{P}|\psi_2\rangle \leftarrow \text{the operator is linear}$$

Hermicity:

$$\langle\psi_1|\hat{P}|\psi_2\rangle = \langle\psi_1|\phi\rangle\langle\phi|\psi_2\rangle = \left[\langle\psi_2|\phi\rangle\langle\phi|\psi_1\rangle\right]^* =$$

$$= \left[\langle\psi_2|\hat{P}\psi_1\rangle\right]^* = \langle\hat{P}\psi_1|\psi_2\rangle \leftarrow \text{the operator is Hermitian}$$

The eigenvalues of $\hat{P}$ are 0 and 1, which can be easily verified by direct substitution (0 corresponds to any state ψ orthogonal to ϕ, while 1 corresponds to ϕ itself):

$$\hat{P}|\psi_\perp\rangle = |\phi\rangle\underbrace{\langle\phi|\psi_\perp\rangle}_{0} = 0|\phi\rangle \qquad \hat{P}|\phi\rangle = |\phi\rangle\langle\phi|\phi\rangle = 1|\phi\rangle$$

Obviously ϕ and all states orthogonal to ϕ form a complete set. Thus, there may be no other eigenvalue/eigenfunction

Lastly

$$\hat{P}^2|\psi\rangle = (|\phi\rangle\langle\phi|)(|\phi\rangle\langle\phi|)|\psi\rangle = |\phi\rangle\langle\phi|\psi\rangle = \hat{P}\psi$$

$$\hat{P}^3|\psi\rangle = (|\phi\rangle\langle\phi|)(|\phi\rangle\langle\phi|)(|\phi\rangle\langle\phi|)|\psi\rangle = |\phi\rangle\langle\phi|\psi\rangle = \hat{P}\psi$$

So $\hat{P}^3 = \hat{P}^2 = \hat{P}$

b)

$$\left.\begin{aligned}\hat{P}\hat{Q}|\psi\rangle &= |\phi\rangle\langle\phi|\chi\rangle\langle\chi|\psi\rangle = \alpha|\phi\rangle \\ \hat{Q}\hat{P}|\psi\rangle &= |\chi\rangle\langle\chi|\phi\rangle\langle\phi|\psi\rangle = \beta|\chi\rangle\end{aligned}\right\}$$

Obviously P and Q do not commute UNLESS $\phi = \chi$ or ϕ is orthogonal to χ ($\langle\phi|\chi\rangle = 0$)

5. $\hat{\vec{S}}^2 = (\hat{\vec{S}}_1 + \hat{\vec{S}}_2)^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$

$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}[\hat{S}^2 - S_1^2 - S_2^2]$ $\quad \hat{H} = \lambda\left(\frac{1}{2}[\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2] + \hbar \hat{S}_z\right)$

Obviously the states $|SMS_1S_2\rangle$ (coupled representation) are eigenstates of this Hamiltonian. In other words, $|SMS_1S_2\rangle$ are eigenstates of S^2, S_1^2, S_2^2, and $\hat{S}_z$ simultaneously. Thus, they are eigenstates of the Hamiltonian.

The energies are then given by

$$E = \lambda\left(\frac{\hbar^2}{2}[S(S+1) - S_1(S_1+1) - S_2(S_2+1)] + \hbar^2 M\right)$$

$S_1(S_1+1) = \frac{3}{2}\left(\frac{3}{2}+1\right) = \frac{15}{4}$ $\qquad S_2(S_2+1) = \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}$

According to the rules of addition of angular momenta S ranges from $|S_1 - S_2|$ to $S_1 + S_2$, i.e. the possible values are $S=1$ and $S=2$. M ranges from $-S$ to $+S$. As a result we have the following energies:

$\underline{S=1}$ $\quad E = \lambda\hbar^2\left(-\frac{5}{4} + M\right)$ or $E = -\frac{9}{4}\lambda\hbar^2$ $\quad E = -\frac{5}{4}\lambda\hbar^2$ $\quad E = -\frac{1}{4}\lambda\hbar^2$

$\underline{S=2}$ $\quad E = \lambda\hbar^2\left(\frac{3}{4} + M\right)$ or $E = -\frac{5}{4}\lambda\hbar^2$ $\quad E = -\frac{1}{4}\lambda\hbar^2$ $\quad E = \frac{3}{4}\lambda\hbar^2$ $\quad E = \frac{7}{4}\lambda\hbar^2$ $\quad E = \frac{11}{4}\lambda\hbar^2$

The energy levels and their degeneracies are:

$-\frac{9}{4}\lambda\hbar^2 \qquad g=1$

$-\frac{5}{4}\lambda\hbar^2 \qquad g=2$

$-\frac{1}{4}\lambda\hbar^2 \qquad g=2$

$\frac{3}{4}\lambda\hbar^2 \qquad g=1$

$\frac{7}{4}\lambda\hbar^2 \qquad g=1$

$\frac{9}{4}\lambda\hbar^2 \qquad g=1$

6. a) If $\ell = 1$ then the eigenvalue of $\hat{L}^2$ is $\ell(\ell+1)\hbar^2$ and the possible eigenvalues of $\hat{L}_z$ are $+\hbar, 0, -\hbar$

b) $s = 1/2$

The eigenvalue of $\hat{S}^2$ is $s(s+1)\hbar^2 = \frac{3}{4}\hbar^2$

The eigenvalues of $\hat{S}_z$ are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

c) According to the rules of addition of angular momenta the eigenvalues of $(\vec{L} + 2\vec{S})_z$ are $-2\hbar, -\hbar, 0, +\hbar, +2\hbar$.

Thus, the eigenvalues of μ_z are $\frac{e\hbar}{m_e}, \frac{e\hbar}{2m_e}, 0, -\frac{e\hbar}{2m_e}, -\frac{e\hbar}{m_e}$

d) If we assume $\vec{B} \parallel \hat{z}$ then $\hat{H} \propto \hat{\mu}_z$

Therefore the beam will split into 5 beams corresponding to different eigenvalues of $\hat{\mu}_z$ (and $\hat{H}$).