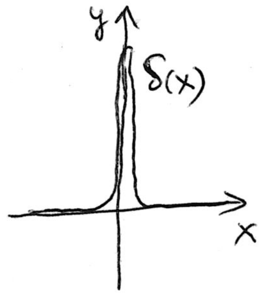
Dirac delta function review

The Dirac delta function is a generalized function that can be defined as the limiting case of a distribution that becomes infinitely narrow and at the same time infinitely tall. Importantly, the transition to the limit must be done in such a way that the area under the curve (the integral) remains constant and equal to 1. In other words, the delta function can be thought as an "impulse" or a "spike" with finite area under the curve that describes it.

From purely mathematical point of view, the delta function is not strictly a function. It only makes sense as a mathematical object when it appears inside an integral. Formally

$$\delta(x) = \begin{cases} \infty & , x = 0 \\ 0 & , x \neq 0 \end{cases} \quad \text{AND} \quad \int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$



The concept of the delta function is convenient when it is necessary to describe distributions that are extremely localized (e.g. charge distribution of a point charge).

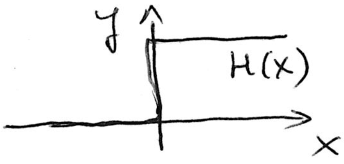
Since the delta function is peaked at a single point only, it is easy to see that for any reasonably nice function $f(x)$ we have

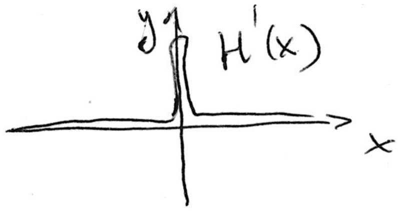
$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0) \quad \text{or} \quad \int_{-\infty}^{+\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

In fact, $\displaystyle\int_{x_0-\epsilon}^{x_0+\epsilon} f(x)\,\delta(x-x_0)\,dx = f(x_0)$ for $\epsilon > 0$

The delta function can also be thought as the derivative of the Heaviside step function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$



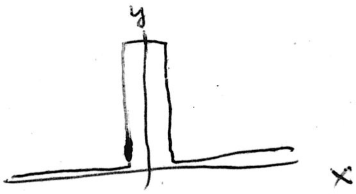


The delta function is a continuous analogue of the Kronecker delta symbol: $\delta(x-y) \Longleftrightarrow \delta_{ij}$

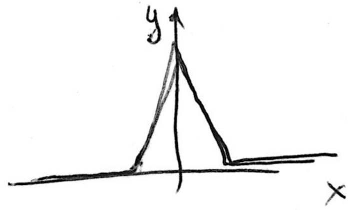
Here are a few examples of limiting sequences that converge to the delta function:

$$\lim_{\epsilon \to 0^+} \frac{1}{\sqrt{\pi}\,\epsilon} e^{-\frac{x^2}{\epsilon^2}} \qquad \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\sin\left(\frac{x}{\epsilon}\right)}{x}$$

$$\lim g_\epsilon(x) \qquad g_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon}, & -\epsilon \leq x \leq \epsilon \\ 0, & \text{otherwise} \end{cases}$$



$$\lim p_\epsilon(x) \qquad p_\epsilon(x) = \begin{cases} \frac{1}{\epsilon^2}(x+\epsilon), & -\epsilon \leq x < 0 \\ \frac{1}{\epsilon^2}(-x+\epsilon), & 0 \leq x \leq \epsilon \\ 0, & \text{otherwise} \end{cases}$$



Let us list the most important properties of the delta function!

1) $\delta(x) = -\delta(x)$

2) $\delta(ax) = \frac{1}{|a|}\delta(x)$

3) $x\delta(x) = 0$

4) $x\delta'(x) = -\delta(x)$

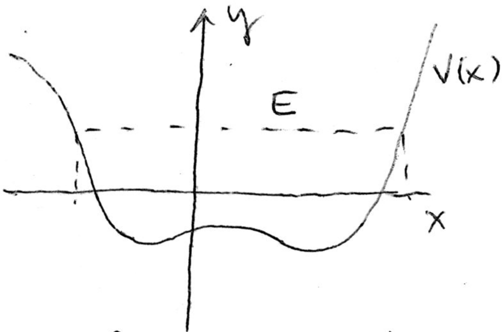
5) $\delta'(-x) = -\delta'(x)$

6) $\displaystyle\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$ where $f'(x)$ is the derivative and x_i are the simple roots of $f(x) = 0$

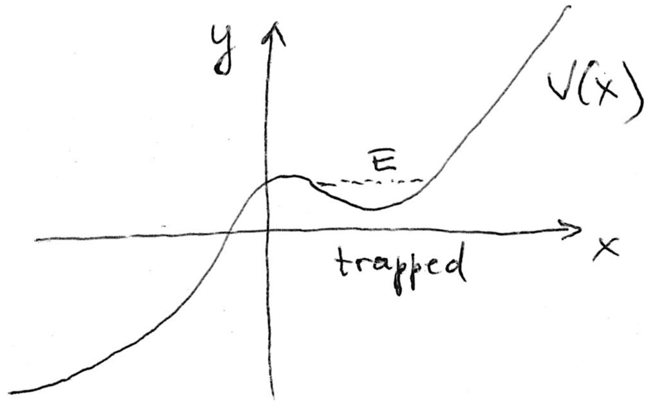
7) $\displaystyle\delta(x-y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(x-y)t}\,dt$

Bound and scattering states in 1D

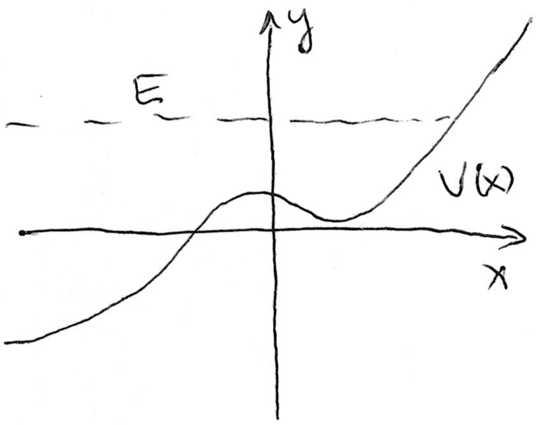
In classical mechanics, depending on the shape of the potential there are three distinct possibilities for the motion of a particle:



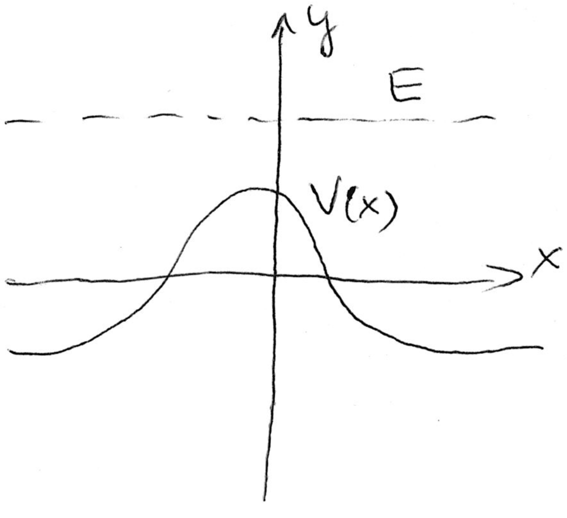
bound state



bound state



scattering



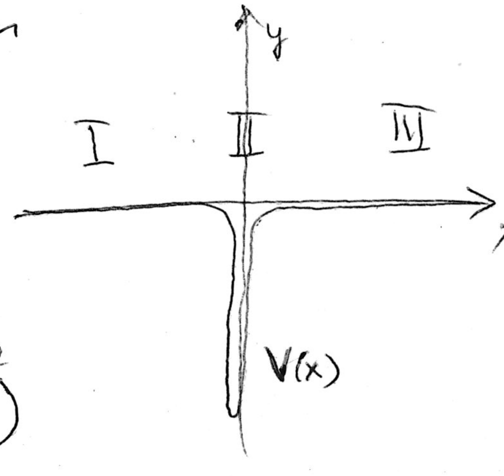
Scattering

Similarly, in quantum mechanics the motion ~~th~~may take different forms.: The solutions to the Schrödinger equation can either correspond to bound states (discrete energy spectrum) or scattering states (continuous spectrum)

Delta function attractive potential

Let us consider the potential $V(x) = -\alpha\delta(x)$ where α is some positive constant.

The SE has the following form



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\delta(x)\psi = E\psi$$

In region I $(x<0)$ it becomes

$$\frac{d^2\psi}{dx^2} = k^2\psi \qquad k = \frac{\sqrt{-2mE}}{\hbar} \qquad \begin{matrix}\text{assume}\\ (E<0)\end{matrix}$$

$$\psi = Ae^{-kx} + Be^{kx}$$

The square integrabity requires that $A=0$, thus

$$\psi = Be^{kx}$$

In region III $(x>0)$ we, in a similar way, obtain

$$\psi = Fe^{-kx}$$

Now we have to "stitch" these two parts of the solution together. ψ is always continuous. ψ' may have discontinuittes at the points where the potential is singular. In our case $B=F$ and

$$\psi(x) = Be^{-k|x|}$$

B can be determined from the normalization condition: $\int_{-\infty}^{+\infty}|\psi|^2dx = 1 \implies B = \sqrt{k}$

But what about k? Is it quantized?

Let us integrate the SE around $x=0$:

$$-\frac{\hbar^2}{2m}\underbrace{\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}dx}_{-\frac{\hbar^2}{2m}\left[\psi'(\epsilon)-\psi'(-\epsilon)\right]} + \underbrace{\int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx}_{-\alpha\psi(0)} = E\underbrace{\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx}_{=0}$$

since ψ is "nice"

$$\psi'(-\epsilon) = \sqrt{k}\cdot k e^{k\epsilon} = k^{3/2}e^{k\epsilon}$$

$$\psi(0) = \sqrt{k}$$

$$\psi'(\epsilon) = \sqrt{k}(-k)e^{-k\epsilon} = -k^{3/2}$$

Therefore, we get the following relation

$$-\frac{\hbar^2}{2m}\left[-k-k\right] = \alpha \quad \Rightarrow \quad \frac{\hbar^2 k}{m} = \alpha \qquad k = \frac{\alpha m}{\hbar^2}$$

What we see is that there is only one value of k allowed for negative energies

$$E = -\frac{m\alpha^2}{2\hbar^2}$$

Interestingly, the number of discrete energy levels is independent of the strength of $V(x)$, i.e. independent of the magnitude of α.

Now let us consider the case when $E > 0$:

$$\frac{d^2\psi}{dx^2} = -k^2\psi \qquad \left(k = \frac{\sqrt{2mE}}{\hbar}\right) \qquad \psi = Ae^{ikx} + Be^{-ikx} \quad x>0$$

$$\psi = Fe^{ikx} + Ge^{-ikx} \qquad x<0$$

The continuity at $x=0$ gives: $F+G = A+B$

$$\psi' = ik\left(Fe^{ikx} - Ge^{-ikx}\right) \quad x>0 \quad \text{or} \quad \psi'(0^+) = ik(F-G)$$

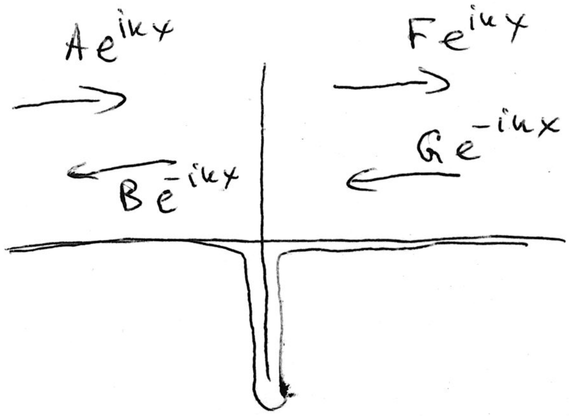
$$\psi' = ik\left(Ae^{ikx} - Be^{ikx}\right) \quad x<0 \quad \text{or} \quad \psi'(0^+) = ik(A-B)$$

Using the integration of the SE again we obtain:

$$ik(F-G-A+B) = -\frac{2m\alpha}{\hbar^2}(A+B)$$

$$F-G = A(1+2i\beta) - B(1-2i\beta) \qquad \beta = \frac{m\alpha}{\hbar^2 k}$$

$\exp(ikx)$ represents a wave propagating to the right and $\exp(-ikx)$ represents a wave propagating to the left. Thus A is the amplitude of the wave coming in from the left, B is the amplitude of the wave returning to the left, F is the amplitude of the wave traveling to the right, and G is the amplitude of the wave coming in from the right.



In case of scattering from the left $G=0$

and

$$B = \frac{i\beta}{1-i\beta}A \qquad F = \frac{1}{1-i\beta}A$$

The relative probability that an incident particle is reflected back is

$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2} \quad \leftarrow \text{reflection coefficient}$$

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2} \quad \leftarrow \text{transmission coefficient}$$

If we change the sign of α we no longer get any bound state ($E<0$) but, interestingly, R and T remain unchanged