In the previous lecture we determined that when the raising/lowering operator acts on an eigen-function of L^2 (which is $|l,m\rangle$) they change the value of m by one:

$$L_\pm |lm\rangle = A_l^m |l; m\pm 1\rangle$$

Here A_l^m is some constant. Let us now find this constant.

We had the following relation:

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

Evaluating $\langle lm| L_\mp L_\pm |lm\rangle$ yielss:

$$\langle lm| L_\mp L_\pm |lm\rangle = \langle lm| L^2 - L_z^2 \mp \hbar L_z |lm\rangle =$$

$$= \langle lm| \hbar^2 l(l+1) - \hbar^2 m^2 \mp \hbar^2 m |lm\rangle =$$

$$= \underbrace{\langle lm|lm\rangle}_{1} \left(\hbar^2 l(l+1) - \hbar^2 m(m\pm 1)\right)$$

On the other hand

$$\langle lm| L_\mp L_\pm |lm\rangle = \langle L_\pm lm| L_\pm lm\rangle = |A_l^m|^2 \langle lm|lm\rangle$$

$$= |A_l^m|^2$$

So

$$A_l^m = \hbar\sqrt{l(l+1) - m(m\pm 1)} \qquad \begin{pmatrix} + \text{ for } L_+ \\ - \text{ for } L_- \end{pmatrix}$$

Matrix representation of the angular momentum operator

We have learned so far that spherical harmonics

$$Y_\ell^m(\theta,\phi) = |\ell, m\rangle$$

are the eigenfunctions of L^2 and L_z operators, such that

$$\langle \ell' m' | L^2 | \ell m\rangle = \hbar^2 \ell(\ell+1)\, \delta_{\ell'\ell}\delta_{m'm}$$

$$\langle \ell' m' | L_z | \ell m\rangle = \hbar m\, \delta_{\ell'\ell}\, \delta_{m'm}$$

If we take Y_ℓ^m as a basis set and order these functions as follows

$$Y_0^0,\ Y_1^1,\ Y_1^0\ Y_1^{-1},\ Y_2^2,\ Y_2^1,\ Y_2^0,\ Y_2^{-1},\ Y_2^{-2},\ \dots$$

then we can write the matrix representation of L^2 as

$$(L^2) = \hbar^2 \begin{pmatrix} \boxed{0} & & & & & & & & & & \\ & 2 & & & & & & & & & \\ & & 2 & & & & & & & & \\ & & & 2 & & & & & & & \\ & & & & 6 & & & & & & \\ & & & & & 6 & & & & & \\ & & & & & & 6 & & & & \\ & & & & & & & 6 & & & \\ & & & & & & & & 6 & & \\ & & & & & & & & & 12 & \\ & & & & & & & & & & \ddots \end{pmatrix}$$

where all the off-diagonal elements are zero. Each dashed "box" above corresponds to the $2\ell+1$ functions associated with the same eigenvalue of L^2.

Similarly, for L_z matrix we have

$$(L_z) = \hbar \begin{pmatrix} 0 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & 0 & & & & & & & \\ & & & -1 & & & & & & \\ & & & & 2 & & & & & \\ & & & & & 1 & & & & \\ & & & & & & 0 & & & \\ & & & & & & & -1 & & \\ & & & & & & & & -2 & \\ & & & & & & & & & 3 \; \ddots \end{pmatrix}$$

Although matrices L^2 and L_z are infinite dimensional, each set of $2\ell+1$ functions associated with a given value of ℓ forms a subspace of dimension $2\ell+1$. Furthermore, the $2\ell+1$ spherical harmonics Y_ℓ^m, $m=-\ell,\dots,\ell$ form a complete, orthonormal basis in that subspace. In other words, any function with carrying a definite value of the angular momentum ℓ can be expanded in terms of Y_ℓ^m $m=-\ell,\dots,\ell$.

The operators $L_\pm$, L_x, and L_y can also be represented by matrices. If we again chose Y_ℓ^m as a basis then

$$L_\pm Y_\ell^m = \hbar\sqrt{\ell(\ell+1)-m(m\pm1)}\; Y_\ell^{m\pm1} \qquad \begin{pmatrix} m+1\le\ell \\ m-1\ge-\ell \end{pmatrix}$$

$$\langle \ell' m' | L_+ | \ell m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m+1)}\;\delta_{\ell'\ell}\,\delta_{m',m+1}$$

$$\langle \ell' m' | L_- | \ell m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m-1)}\;\delta_{\ell'\ell}\,\delta_{m',m-1}$$

For example, in the subspace of $\ell=1$, $L_\pm$ are the following 3×3 matrices

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

(btw we can easily see that $L_+^\dagger = L_-$)

In this matrix representation, the eigenfunctions Y_ℓ^m become unit column vectors with $2\ell+1$ components:

$$Y_1^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad Y_1^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad Y_1^{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We can verify by matrix-vector multiplication that

$$L_+ Y_1^1 = 0 \qquad L_+ Y_1^0 = \hbar\sqrt{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hbar\sqrt{2}\, Y_1^1$$

$$L_+ Y_1^{-1} = \hbar\sqrt{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hbar\sqrt{2}\, Y_{10}, \quad \text{etc.}$$

In the subspace with $\ell=1$ the matrices for L_x and L_y are

$$L_x = \frac{1}{2}(L_+ + L_-) = \hbar \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$L_y = \frac{1}{2i}(L_+ - L_-) = \hbar \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

In the previous lecture we saw that the commutation relations $[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$ can be satisfied not only for integer values of l, but also for half-integer values of the quantum number. In this case there are no spherical harmonics.

The case where the angular momentum quantum number is $1/2$ is of particular interest. It represents the intrinsic angular momentum, or spin, of many elementary particles, such as electrons, protons, neutrons, and quarks. Traditionally, for spin, the angular momentum quantum number is denoted by s:

$$S^2 \psi_{sm} = \hbar^2 s(s+1) \psi_{sm} \qquad S_z \psi_{sm} = \hbar m \psi_{sm}$$

where ψ_{sm} is a spin wave function to be determined

$$s(s+1) = \tfrac{1}{2}\left(\tfrac{1}{2}+1\right) = \tfrac{3}{4} \qquad m = \pm 1/2$$

Let $\psi_{\frac{1}{2},\frac{1}{2}} \equiv \psi_\uparrow$ and $\psi_{\frac{1}{2},\frac{1}{2}} = \psi_\downarrow$

$$S_z \psi_\uparrow = \frac{\hbar}{2} \psi_\uparrow \qquad S_z \psi_\downarrow = -\frac{\hbar}{2} \psi_\downarrow$$

$$S_+ \psi_\downarrow = \hbar \sqrt{s(s+1) - m(m+1)}\, \psi_\uparrow = \hbar \sqrt{\tfrac{3}{4} - \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}+1\right)}\, \psi_\uparrow = \hbar \psi_\uparrow$$

$$S_- \psi_\uparrow = \hbar \sqrt{s(s+1) - m(m-1)}\, \psi_\downarrow = \hbar \sqrt{\tfrac{3}{4} - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}-1\right)}\, \psi_\downarrow = \hbar \psi_\downarrow$$

Matrix representation for $s = 1/2$ is

$$S^2 = \hbar^2 \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \qquad S_z = \hbar \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

$$S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\psi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

In the literature one can often encounter the Pauli matrices, which are defined as

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Pauli matrices have some simple properties:

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k \qquad \sigma_i^2 = \hat{1}$$

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\hat{1} \quad \text{(anticommutation)}$$

Eigenvalues and eigenvectors of σ_i can be easily determined

For example, for σ_x we have

$$\sigma_x \psi = \lambda \psi \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$0 = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 \quad \Rightarrow \quad \lambda = \pm 1$$

$$\begin{pmatrix} \mp 1 & 1 \\ 1 & \mp 1 \end{pmatrix} \begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{aligned} \mp a_\pm + b_\pm &= 0 \\ b_\pm &= \pm a_\pm \end{aligned}$$

$$\psi_{s_x = \pm \frac{\hbar}{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(\psi_\uparrow \pm \psi_\downarrow \right)$$