

Spin

In classical mechanics angular motion can be decomposed into the orbital part ($\vec{L} = \vec{r} \times \vec{p}$) associated with the motion of the center of mass, and spin ($\vec{S} = I\vec{\omega}$) associated with the motion about the center of mass. Something analogous takes place in quantum mechanics. The difference, however, is that spin motion is intrinsic and has nothing to do with motion in space. Spin is thus an intrinsic property of all elementary particles.

The idea of spin was introduced in physics in 1925 by Uhlenbeck and Goudsmit, who came to realize about the existence of spin based on the analysis of the experimental facts. In particular, the existence of spin followed from the experiment of Stern and Gerlach, where a splitting of a beam of atoms occurs. For example, both orbital and magnetic moments of an electron in the ground state of hydrogen equal to zero. At the same time a beam of H-atoms split in nonhomogeneous magnetic field ^{into two}, indicating two possible projections of the intrinsic magnetic moment. The experiments had shown that the absolute value of this moment is equal to μ_B :

$$\mu_B = \frac{e\hbar}{2m_e c} = \underbrace{\frac{e\hbar}{2m_e c}}_{\text{SI units}}$$

Gaussian units

The hypothesis of spin allows to explain the experimental observations in a simple way

The algebraic theory of spin is the same as that of the orbital angular momentum and is based on the fundamental commutation relations:

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z \quad \text{or more generally} \quad [\hat{S}_i, \hat{S}_j] = i\hbar \epsilon_{ijk} \hat{S}_k$$

The eigenvectors and eigenvalues of S^2 and S_z are

$$\hat{S}^2 |sm\rangle = \hbar^2 s(s+1) |sm\rangle \quad \hat{S}_z |sm\rangle = \hbar m |sm\rangle$$

Lastly the action of the raising/lowering operator is

$$\hat{S}_{\pm} |sm\rangle = \hbar \sqrt{s(s+1) - m(m\pm 1)} |s(m\pm 1)\rangle \quad \hat{S}_{\pm} = \hat{S}_x \pm i\hat{S}_y$$

s may take both integer and half-integer values

While all particles possess spin ($0, \frac{1}{2}$, or something else) the most important case from practical point of view is $s = 1/2$. Many particles have $s = 1/2$. Also, this is the simplest case for the theoretical

consideration. All spin operators in case $s = 1/2$ can be represented by 2×2 matrices, while the eigenstates are two-component objects called spinors.

$$\left. \begin{aligned} |s = \frac{1}{2}, m = \frac{1}{2}\rangle &\equiv |\frac{1}{2}, \frac{1}{2}\rangle \equiv |\uparrow\rangle \equiv |\alpha\rangle \\ |s = \frac{1}{2}, m = -\frac{1}{2}\rangle &= |\frac{1}{2}, -\frac{1}{2}\rangle = |\downarrow\rangle \equiv |\beta\rangle \end{aligned} \right\} \begin{array}{l} \text{usual notations} \\ \text{for spin eigenstates} \end{array}$$

A general state, χ , can be written as

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a|\uparrow\rangle + b|\downarrow\rangle \quad |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For \hat{S}^2 matrix we have:

$$\begin{pmatrix} S_{11}^2 & S_{12}^2 \\ S_{21}^2 & S_{22}^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4} \hbar^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} S_{11}^2 & S_{12}^2 \\ S_{21}^2 & S_{22}^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4} \hbar^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solving for S_{ij} yields the following

$$\hat{S}_z = \frac{3}{4} \hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly

$$\begin{pmatrix} S_{z11} & S_{z12} \\ S_{z21} & S_{z22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} S_{z11} & S_{z12} \\ S_{z21} & S_{z22} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Lastly,

$$\hat{S}_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \hat{S}_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \hat{S}_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{S}_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

and

$$\hat{S}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \hat{S}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Since $\hat{S}_{\pm} = \hat{S}_x \pm i\hat{S}_y$
we get the explicit matrix form of \hat{S}_x and \hat{S}_y

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

For simplicity of notations Pauli spin matrices are often adopted:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ (which stand for $\hat{S}_x = \frac{\hbar}{2} \hat{\sigma}_x$, $\hat{S}_y = \frac{\hbar}{2} \hat{\sigma}_y$, $\hat{S}_z = \frac{\hbar}{2} \hat{\sigma}_z$)

For a general state $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ $|a|^2$ gives the probability of observing $m = \frac{\hbar}{2}$, while $|b|^2$ gives the probability of observing $m = -\frac{\hbar}{2}$ state. $|a|^2 + |b|^2 = 1$