Properties of the Pauli matrices.

As the operators corresponding to different components of spin obey the fundamental commutation relatoons,

$$[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk} \hbar \hat{S}_k$$

so do the Pauli matrices (recall that $\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma}$):

$$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\,\varepsilon_{ijk}\,\hat{\sigma}_k$$

The eigenvalues of $\hat{\sigma}_i$ are ± 1. The eigenvectors of σ_z that correspond to $\pm\frac{\hbar}{2}$ projection of spin on the z-axis are

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The eigenvectors of σ_x and σ_y can be expressed as a linear combination of $|\uparrow\rangle$ and $|\downarrow\rangle$

In addition of the above commutation relations, the Pauli matrices posess some other important properties.

Since the eigenvalues of $\hat{\sigma}_i$ are ± 1, the eigenvalues of $\hat{\sigma}_i^2$ are $+1$ (twice degenerate). In their own basis $\hat{\sigma}_i^2$ are the identity matrices

$$\hat{\sigma}_i^2 = \hat{1}$$

(recall that the identity matrix remains the identity matrix in any representation)

Indeed, this can be easily verified using the explicit matrix form:

$$\hat{\sigma}_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{\sigma}_y^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\hat{\sigma}_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore in any representation $\hat{\sigma}_i^2 = \hat{1}$.

Using the commutation relations and the fact that $\hat{\sigma}_i^2 = \hat{1}$ we can show that $\hat{\sigma}_i$ and $\hat{\sigma}_j$ anticommute:

$$\hat{\sigma}_i \hat{\sigma}_j - \hat{\sigma}_j \hat{\sigma}_i = 2i\,\varepsilon_{ijk}\,\hat{\sigma}_k$$

Multiplying by $\hat{\sigma}_i$ from the left/right yields:

$$\underbrace{\hat{\sigma}_i^2}_{1} \hat{\sigma}_j - \hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_i = 2i\,\varepsilon_{ijk}\,\hat{\sigma}_i \hat{\sigma}_k$$

$$\hat{\sigma}_i \hat{\sigma}_j \hat{\sigma}_i - \hat{\sigma}_j \underbrace{\hat{\sigma}_i^2}_{1} = 2i\,\varepsilon_{ijk}\,\hat{\sigma}_k \hat{\sigma}_i$$

When we add the two lines we get:

$$0 = 2i\,\varepsilon_{ijk}\left(\hat{\sigma}_i \hat{\sigma}_k + \hat{\sigma}_k \hat{\sigma}_i\right) \qquad i \neq k$$

Thus,

$$\{\hat{\sigma}_i, \hat{\sigma}_k\} \equiv \hat{\sigma}_i \hat{\sigma}_k + \hat{\sigma}_k \hat{\sigma}_i = 0 \qquad \text{when } i \neq k$$

Or, more generally,

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\,\delta_{ij}\,\hat{1}$$

We can also see that

$$\hat{\sigma}_i \hat{\sigma}_j = \delta_{ij}\,\hat{1} + i\,\varepsilon_{ijk}\,\hat{\sigma}_k$$

This is because

$$\sigma_j \sigma_i + \sigma_i \sigma_j = 2\,\delta_{ij}\,\hat{1}$$

$$\underbrace{\sigma_j \sigma_i - \sigma_i \sigma_j}_{-2i\,\varepsilon_{ijk}\,\sigma_k} + \underbrace{\sigma_i \sigma_j + \sigma_i \sigma_j}_{2\sigma_i \sigma_j} = 2\,\delta_{ij}\,1$$

$$\sigma_i \sigma_j = \delta_{ij}\,\hat{1} + i\,\varepsilon_{ijk}\,\hat{\sigma}_k$$

The importance of the latter formula lies in the fact that any product of σ_i operators, $\sigma_i \sigma_j \dots \sigma_k$, can be linearized.

Let us note that the commutation relations $[\hat{S}_i, \hat{S}_j] = i \varepsilon_{ijk} \hat{S}_k$ hold true for any value of spin/angular momentum, while the anticommutation relations $\{\sigma_i, \sigma_k\} = 2\delta_{ik} \hat{1}$ take place for the case of $S = \frac{1}{2}$ only.

The possibility to linearize any product $\sigma_i \dots \sigma_k$ implies that any function of $a\hat{1} + \vec{b}\cdot\vec{\sigma}$ can be written in terms of $\hat{1}$ and $\vec{b}\vec{\sigma}$, where $\vec{b}$ is an arbitrary vector. This is because any "nice" function can be expressed through its Taylor series. Thus,

$$F(a\hat{1} + \vec{b}\vec{\sigma}) = A\hat{1} + B\,\vec{n}\cdot\vec{\sigma}$$

where A and B are constants

This can also be realized if we recall that σ_i and $\hat{1}$ are 2×2 matrices. Thus, four of them form a "complete set" of matrices. Any 2×2 matrix (including any function of a 2×2 matrix) can be expressed as a linear combination of $\hat{1}$ and $\hat{\sigma}_i$'s.

Electron in magnetic field

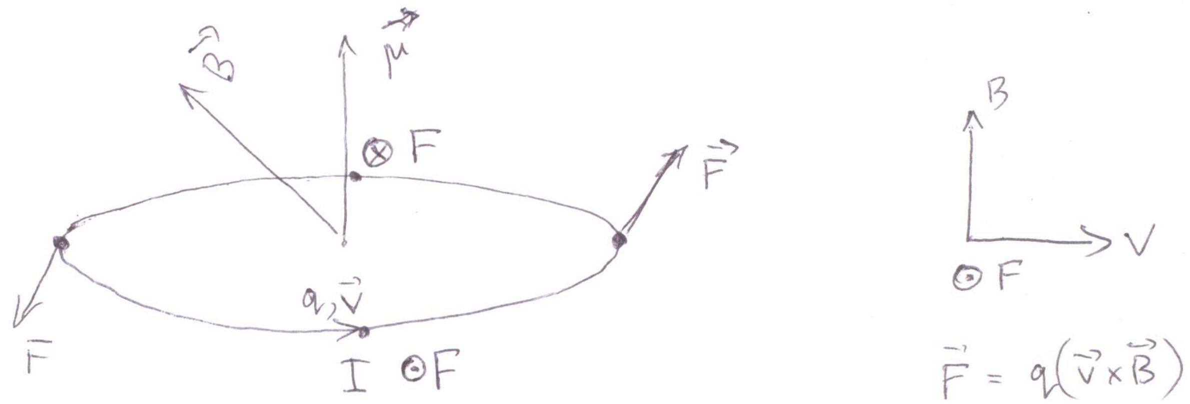
In classical mechanics a spinning charged particle forms a magnetic dipole. This magnetic dipole is proportional to to its spin angular momentum:

$$\vec{\mu} = \gamma \vec{S}$$

where γ is a constant that depends on the magnitude and sign of the charge.

A similar relationship takes place in quantum mechanics. Constant γ is called the gyromagnetic ratio and is equal $\gamma = -\frac{e}{m}$ (SI or Gauss)

When a magnetic dipole is placed in a magnetic field $\vec{B}$, it experiences a torque, $\vec{\mu} \times \vec{B}$, which tends to line it up parallel to the $\vec{B}$ field.



The energy associated with this torque is $H = -\vec{\mu} \cdot \vec{B}$. Hence, the Hamiltonian of a particle with spin in a magnetic field becomes

$$\hat{H} = -\gamma \vec{B} \cdot \vec{S}$$

Larmor precession

Consider a particle of spin 1/2 at rest in a uniform magnetic field, which points in the z-direction

$$\vec{B} = B\,\vec{e}$$

The interaction of the particle with this field is described by the Hamiltonian

$$\hat{H} = -\gamma \vec{B}\cdot\hat{\vec{S}} = -\gamma B \hat{S}_z = -\frac{\gamma B \hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

The eigenstates of $\hat{H}$ are the same as those of $\hat{S}_z$:

$$E_+ = -\frac{\gamma B \hbar}{2} \qquad \chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$$

$$E_- = \frac{\gamma B \hbar}{2} \qquad \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

Energy is lowest when the dipole is parallel to the $\vec{B}$ field (i.e. when its projection on the $\vec{B}$ axis is positive)

Now let us see how the spin state of the particle evolves with time. The general solution of the time-dependent Schrödinger equation

$$i\hbar\frac{\partial \chi}{\partial t} = \hat{H}\chi \qquad \chi \equiv \begin{pmatrix}f(t)\\ g(t)\end{pmatrix}$$

can be expressed in terms of the stationary states:

$$\chi(t) = a\chi_+ e^{-\frac{iE_+t}{\hbar}} + b\chi_- e^{-\frac{iE_-t}{\hbar}} = \begin{pmatrix}a e^{\frac{i\gamma B t}{2}}\\ b e^{-\frac{i\gamma B t}{2}}\end{pmatrix}$$

at $t=0$

$$\chi(0) = \begin{pmatrix}a\\b\end{pmatrix} \qquad \text{and} \qquad |a|^2+|b|^2 = 1$$

We can write a and b as

$$a = \cos\frac{\alpha}{2} \qquad b = \sin\frac{\alpha}{2} \qquad \left(\text{so that } |a|^2+|b|^2=1\right)$$

Then

$$\chi(t) = \begin{pmatrix} \cos\frac{\alpha}{2}\, e^{\frac{i\gamma B t}{2}} \\ \sin\frac{\alpha}{2}\, e^{-\frac{i\gamma B t}{2}} \end{pmatrix}$$

Now let us compute $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$

$$\langle S_x \rangle = \chi^{+}(t)\, S_x\, \chi(t) = \left(\cos\frac{\alpha}{2} e^{-\frac{i\gamma B t}{2}},\ \sin\frac{\alpha}{2} e^{\frac{i\gamma B t}{2}}\right) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} e^{\frac{i\gamma B t}{2}} \\ \sin\frac{\alpha}{2} e^{-\frac{i\gamma B t}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{2} \sin\alpha \cos(\gamma B t)$$

Similarly

$$\langle S_y \rangle = \chi^{+}(t)\, S_y\, \chi(t) = -\frac{\hbar}{2} \sin\alpha \sin(\gamma B t)$$

Lastly

$$\langle S_z \rangle = \chi^{+}(t)\, S_z\, \chi(t) = \frac{\hbar}{2}\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) = \frac{\hbar}{2}\cos\alpha$$

We can see that $\langle \vec{S} \rangle$ precesses about the z-axis ($\vec{B}$ direction) with the Larmor frequency:

$$\omega = \gamma B$$