

Stern - Gerlach experiment

The energy associated with a magnetic dipole in magnetic field is $-\vec{\mu} \cdot \vec{B}$. If $\vec{\mu}$ is not aligned with the direction of \vec{B} then there is a torque that tries to align $\vec{\mu}$ with \vec{B} . If \vec{B} is uniform then the net force vanishes because $\vec{F} = -\frac{\partial V}{\partial \vec{r}} = 0$ ($V = -\vec{\mu} \cdot \vec{B}$). In inhomogeneous magnetic field, however, this force is no longer zero. In fact, it can be used to separate out particles with a particular spin orientation.

Let us assume that we have a situation where heavy neutral atoms traveling in y-direction enter a region of weak inhomogeneity so that the field is not $B \hat{e}_z$ but $\vec{B}(x, y, z) = -\alpha x \hat{e}_x + (B + \alpha z) \hat{e}_z$, where α is small. The distortion just along z-axis is impossible because $\nabla \cdot \vec{B} = 0$. For this reason we have to have the field distorted along both x and z axes.

The force is then

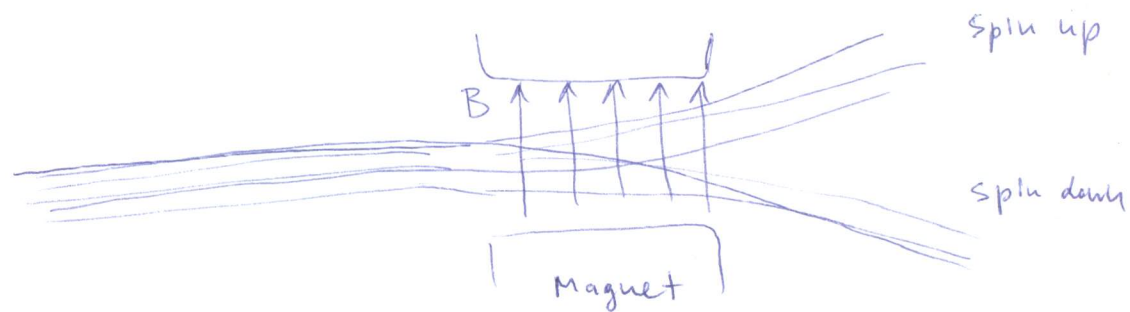
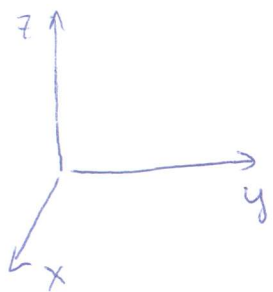
$$\begin{aligned} \vec{F} &= -\nabla(-\vec{\mu} \cdot \vec{B}) = -\nabla(-\gamma \vec{S} \cdot \vec{B}) \\ &= \gamma \alpha (-S_x \hat{e}_x + S_z \hat{e}_z) \end{aligned}$$

Now from the previous lecture we know that the x-component of spin oscillates rapidly (with frequency $\omega = \gamma B$) and averages to zero. The z-component, F_z , does not vanish because $\langle S_z \rangle$ is constant (i.e. does not oscillate):

$$F_z = \gamma \alpha S_z$$

Therefore, particles are pulled up or down (in the z -direction) depending on the z -component of spin. For classical particles we would observe a smear (because S_z can take any value from $-S$ to $+S$). For quantum particles, however, S_z is quantized. In fact the beam of particles with angular momentum S splits into $2s+1$ separate streams.

In case of atoms this splitting effect comes mainly from unpaired electron(s). This is because it is proportional to $\gamma = \frac{q}{m}$ and the mass of protons and neutrons exceeds that of electrons by orders of magnitude. For hydrogen and alkali atoms the beam splits into just two streams



We can also show that a beam of spin particles is split indeed in a little more rigorous way that does not invoke classical concepts (e.g. force). Consider the process in a reference frame that moves along the y -axis with the beam. In this frame the Hamiltonian starts out zero, turns on for time T (time necessary to pass the magnet), and then turns off again:

$$H(t) = \begin{cases} 0 & t < 0 \\ -\gamma(B + dz)S_z & 0 \leq t \leq T \\ 0 & t > T \end{cases}$$

above we ignored the x-component of B because of the reasons outlined previously.

Suppose an atom with a single electron is in state at $t=0$

$$\chi(t_0) = a\chi_+ + b\chi_-$$

While the Hamiltonian acts, $\chi(t)$, evolves in the usual way (from $t=0$ to $t=T$ \hat{H} is time-independent)

$$\chi(t) = a\chi_+ e^{-\frac{iE_+t}{\hbar}} + b\chi_- e^{-\frac{iE_-t}{\hbar}} \quad 0 < t < T$$

where $E_{\pm} = \mp \gamma (B + \alpha z) \frac{\hbar}{2}$ (recall that $H = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2} \gamma (B + \alpha z)$)

at $t=T$ it becomes

$$\chi(t=T) = a\chi_+ e^{i\frac{\gamma T B}{2}} e^{i\frac{\alpha \gamma T}{2} z} + b\chi_- e^{-i\frac{\gamma T B}{2}} e^{-i\frac{\alpha \gamma T}{2} z}$$

We can now see that each term carry momentum in the z-direction. Indeed the plane waves are eigenfunctions of the \hat{p}_z operator corresponding to the eigenvalue

$$p_z = \pm \frac{\alpha \gamma T \hbar}{2}$$

Hence, after exiting the magnet, the spin-up component acquires a momentum in the positive z-direction while the spin-down component acquires a momentum in the opposite direction.

The Stern-Gerlach experiment played an important role for the development of quantum theory. It demonstrated that particles possess an intrinsic angular momentum and this momentum takes only discrete projections on any axis.