Problem 1

a) Yes, the state corresponds to a definite energy because the time-dependence is in the form compatible with stationary states:

$$e^{-\frac{iCt}{\hbar}} \qquad E = C$$

b) for $x<0$ $\psi = 0$ $\Rightarrow$ $V(x) = \infty$

for $x>0$ we plug ψ into the Schrödinger equation. In fact, we can take just the spatial part of $\psi(x,t)$ (let us call it φ) and plug it into the stationary Schrödinger equation:

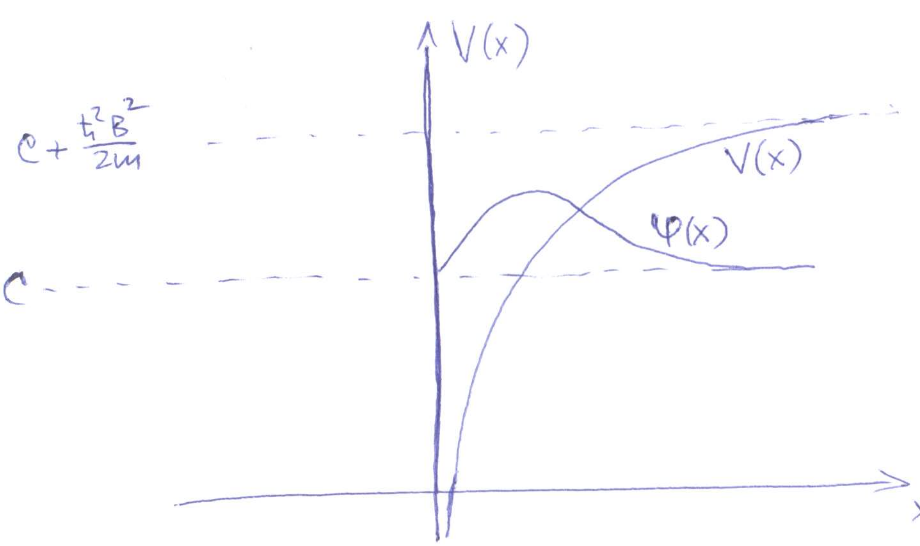
$$\varphi(x) = Axe^{-Bx} \quad (x>0) \qquad H\varphi = C\varphi$$

$$\frac{d\varphi}{dx} = Ae^{-Bx} - ABxe^{-Bx} \qquad \frac{d^2\varphi}{dx^2} = -ABe^{-Bx} - ABe^{-Bx} + AB^2xe^{-Bx} =$$

$$= (-2B + B^2x)Ae^{-Bx} = \left(-\frac{2B}{x} + B^2\right)\varphi$$

$$-\frac{\hbar^2}{2m}\left(-\frac{2B}{x} + B^2\right)\cancel{\varphi} + V\cancel{\varphi} = C\cancel{\varphi}$$

$$V = C + \frac{\hbar^2}{2m}\left(B^2 - \frac{2B}{x}\right) = \left(C + \frac{\hbar^2B^2}{2m}\right) - \frac{2\hbar^2B}{2m}\frac{1}{x} \qquad (x>0)$$



Problem 2 We can rearrange $V(x)$ is such a way that the linear term is eliminated:

$$V(x) = \frac{m\omega^2 x^2}{2} - bx = \frac{m\omega^2}{2}\left[x^2 - \frac{2b}{m\omega^2}x\right] = \frac{m\omega^2}{2}\left[\left(x - \frac{b}{m\omega^2}\right)^2 - \frac{b^2}{m^2\omega^4}\right] =$$

$$= \frac{m\omega^2}{2}(x - x_e)^2 + V_e$$

where $x_e = \frac{b}{m\omega^2}$ and $V_e = -\frac{b^2}{2m\omega^2}$

This potential is essentially the same as the ordinary harmonic oscillator potential. The only difference is that the energy scale is shifted by V_e and spatially the potential is shifted by x_e (which does not affect the energy). The solution of the Schrödinger equation is then obviously

$$E_n = V_e + \hbar\omega\left(n + \tfrac{1}{2}\right)$$

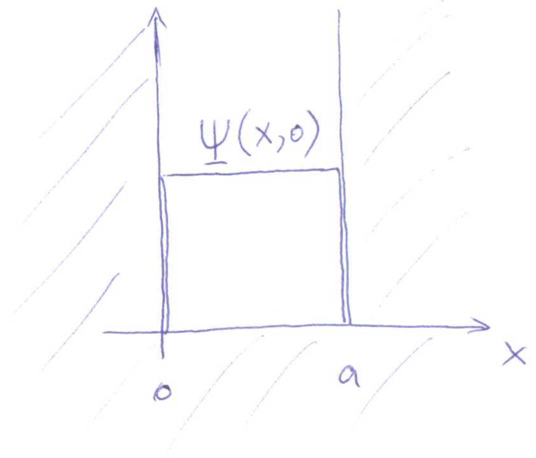
In particular the ground state energy is

$$E_0 = V_0 + \frac{\hbar\omega}{2} = -\frac{b^2}{2m\omega^2} + \frac{\hbar\omega}{2}$$

The ground state wave function is (see formula sheet)

$$\psi(x) = \frac{\alpha^{1/4}}{\pi^{1/4}} e^{-\frac{\alpha(x - x_e)^2}{2}} \qquad \alpha = \frac{m\omega}{\hbar}$$

Problem 3



a) $$1 = \int_0^a |\Psi(x,0)|^2 dx = \int_0^a A^2 dx = A^2 a$$

$$A = \frac{1}{\sqrt{a}}$$

Thus $$\Psi(x,0) = \begin{cases} \frac{1}{\sqrt{a}}, & 0<x<a \\ 0, & \text{otherwise} \end{cases}$$

b) $$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-\frac{iE_n t}{\hbar}}$$ where $$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$ $$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$c_n = \int_0^a \psi_n^*(x)\, \Psi(x,0)\, dx = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \frac{1}{\sqrt{a}} dx =$$

$$= -\frac{\sqrt{2}}{a} \frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\Big|_0^a = -\frac{\sqrt{2}}{\pi n}\Big[\underbrace{\cos(n\pi)}_{(-1)^n} - 1\Big] = \begin{cases} 0, & n \text{ is even} \\ \frac{\sqrt{8}}{\pi n}, & n \text{ is odd} \end{cases}$$

so

$$\Psi_n(x,t) = \sum_{k=0}^{\infty} \frac{4}{\pi\sqrt{a}\,(2k+1)} \sin\left(\frac{(2k+1)\pi x}{a}\right) e^{-i\frac{(2k+1)^2\pi^2\hbar}{2ma^2}t}$$

Problem 4

$$H = \frac{p^2}{2m} + a|x|$$

We can try to minimize the energy

$$E = \langle H \rangle = \frac{\langle p^2 \rangle}{2m} + \alpha \langle |x| \rangle$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \implies \langle p^2 \rangle \geq (\Delta p)^2$$

So we can use $(\Delta p)^2$ instead of $\langle p^2 \rangle$ as a lower bound in the expression for E

Now $\min \langle |x| \rangle \gtrsim \Delta x$

then

$$E \geq \frac{(\Delta p)^2}{2m} + a \Delta x$$

From the uncertainty principle we know $\Delta p \Delta x \geq \frac{\hbar}{2}$ or $\Delta x \approx \frac{\hbar}{2\Delta p}$

Hence, $E \gtrsim \frac{(\Delta p)^2}{2m} + \frac{a\hbar}{2\Delta p}$

The minimum of the above expression is achieved when

$$\frac{\partial E}{\partial \Delta p} = 0 \implies \frac{\Delta p}{m} - \frac{a\hbar}{2(\Delta p)^2} = 0 \implies (\Delta p)^3 = \frac{a\hbar m}{2}$$

$$\Delta p = \left(\frac{a\hbar m}{2}\right)^{1/3}$$

$$E \geq \left(\frac{a\hbar m}{2}\right)^{2/3} \frac{1}{2m} + \frac{a\hbar}{2}\left(\frac{2}{a\hbar m}\right)^{1/3} = \frac{a^{2/3}\hbar^{2/3}}{m^{1/3}}\left(\frac{1}{2^{5/3}} + \frac{1}{2^{2/3}}\right) =$$

$$= \frac{3}{2^{5/3}} \frac{a^{2/3}\hbar^{2/3}}{m^{1/3}} = \frac{3}{2m}\left(\frac{a\hbar m}{2}\right)^{2/3}$$