1. $\psi(r) = R_{10}(r)\, Y_{00}(\theta,\varphi) = \frac{2}{a^{3/2}} e^{-\frac{r}{a}} \frac{1}{\sqrt{4\pi}}$

$|\psi(r)|^2 = \frac{1}{\pi a^3} e^{-\frac{2r}{a}}$ $\qquad x = r\sin\theta\cos\varphi \qquad x^2 = r^2\sin^2\theta\cos^2\varphi$

a) $\langle x\rangle = 0$ because of the symmetry of the integrand

$$\langle x^2\rangle = \frac{1}{\pi a^3} \underbrace{\int_0^{2\pi} \cos^2\varphi\, d\varphi}_{\int_0^{2\pi}\frac{1-\sin 2\varphi}{2}d\varphi = \pi} \cdot \underbrace{\int_0^{\pi} \sin^3\theta\, d\theta}_{\frac{4}{3} \text{ (see formula Sheet)}} \cdot \underbrace{\int_0^{\infty} r^4 e^{-\frac{2r}{a}} dr}_{\frac{4!}{\left(\frac{2}{a}\right)^5} \text{ (see formula sheet)}} =$$

$$= \frac{4}{3a^3}\cdot\frac{24a^5}{32} = a^2$$

$$\langle r\rangle = \frac{1}{\pi a^3} \underbrace{\int_0^{2\pi} d\varphi \int_0^{\pi}\sin\theta\, d\theta}_{4\pi} \underbrace{\int_0^{\infty} r^3 e^{-\frac{2r}{a}} dr}_{\frac{3!}{\left(\frac{2}{a}\right)^4}} = \frac{4}{a^3}\,\frac{6a^4}{16} = \frac{3}{2}a$$

$$\langle r^2\rangle = \frac{1}{\pi a^3}\cdot 4\pi\cdot\frac{24a^5}{32} = 3a^2$$

b) $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = a$ $\qquad \Delta r = \sqrt{\langle r^2\rangle - \langle r\rangle^2} = \sqrt{3a^2 - \frac{9}{4}a^2} = \frac{\sqrt{3}}{2}a$

c) $$P(0<r<\Delta r) = \frac{1}{\pi a^3}\cdot 4\pi\cdot\int_0^{\Delta r} r^2 e^{-\frac{2r}{a}} dr = \frac{4}{a^3}\int_0^{\Delta r} r^2 e^{-\frac{2r}{a}} dr$$

Let us compute the following integral:

$$\int_0^b x^2 e^{-\alpha x} dx = \frac{\partial^2}{\partial\alpha^2}\int_0^b e^{-\alpha x} dx = \frac{\partial^2}{\partial\alpha^2}\frac{1}{\alpha}\cdot\left(1-e^{-\alpha b}\right) = \frac{2}{\alpha^3}\left(1-e^{-\alpha b}\right) - \frac{2b e^{-\alpha b}}{\alpha^2} - \frac{b^2 e^{-\alpha b}}{\alpha}$$

With that we have: $$P = \frac{4}{a^3}\left[\frac{2}{\left(\frac{2}{a}\right)^3}\left(1-e^{-\sqrt{3}}\right) - \frac{2\frac{\sqrt{3}}{2}a e^{-\sqrt{3}}}{\left(\frac{2}{a}\right)^2} - \frac{\frac{3}{4}a^2 e^{-\sqrt{3}}}{\frac{2}{a}}\right] =$$

$$= \left(1-e^{-\sqrt{3}}\right) - \sqrt{3}e^{-\sqrt{3}} - \frac{3}{2}e^{-\sqrt{3}} = 1 - \frac{5+2\sqrt{3}}{2}e^{-\sqrt{3}} \approx 0.251$$

2. $\psi_1 = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2$ or $\phi_1 = \frac{3}{5}\psi_1 + \frac{4}{5}\psi_2$

$\psi_2 = \frac{4}{5}\phi_1 - \frac{3}{5}\phi_2$ $\phi_2 = \frac{4}{5}\psi_1 - \frac{3}{5}\psi_2$

a) Upon measurement that yields a_1 the wave function collapses to ψ_1

b) The possible results of B measurement are:

b_1: $P_{b_1} = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$

b_2: $P_{b_2} = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$

c) $P_{a_1} = P_{b_1} \cdot \left(\frac{3}{5}\right)^2 + P_{b_2} \cdot \left(\frac{4}{5}\right)^2 = \frac{9}{25} \cdot \frac{9}{25} + \frac{16}{25} \cdot \frac{16}{25} = \frac{337}{625}$

3. a) Looking at $l=1$ spherical harmonics in cartesian coordinates (see formula sheet) one can easily see that $(x+y+3z) = \left(C_1^{-1}Y_1^{-1} + C_1^{1}Y_1^{1} + C_1^{0}Y_1^{0}\right) R(r)$ where C_l^m are some coefficients and $R(r)$ is spherically symmetric radial part. This allows us to say right away that $\psi(r)$ is an eigenfunction of $\hat{L}^2$ with eigenvalue $l=1$. However, $\psi(r)$ is not an eigenfunction of $\hat{L}_z$

b) Here we need to expand $x+y+3z$ in terms of spherical harmonics and find C_l^m:

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \qquad Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\frac{x-iy}{r} \qquad Y_1^1 = -\sqrt{\frac{3}{8\pi}}\frac{x+iy}{r}$$

$$z = \sqrt{\frac{4\pi}{3}}\, r\, Y_1^0 \qquad x = \sqrt{\frac{2\pi}{3}}\, r\left(Y_1^{-1} - Y_1^1\right) \qquad y = i\sqrt{\frac{2\pi}{3}}\, r\left(Y_1^{-1} + Y_1^1\right)$$

$$(x+y+3z)f(r) = \sqrt{\frac{2\pi}{3}}\, r\left[Y_1^{-1} - Y_1^1 + iY_1^{-1} + iY_1^1 + 3\sqrt{2}\,Y_1^0\right] f(r) =$$

$= \left[(1+i)Y_1^{-1} + (-1+i)Y_1^1 + 3\sqrt{2}\,Y_1^0\right] g(r)$. where $g(r)$ is some spherically symmetric expression. If we normalize the expression in the square brackets we will get:

$$(1-i)(1+i) + (-1-i)(-1+i) + (3\sqrt{2})^2 = 22$$

So

$$(x+y+3z)f(r) = \underbrace{\left[\frac{1+i}{\sqrt{22}}Y_1^{-1} + \frac{-1+i}{\sqrt{22}}Y_1^1 + \frac{3}{\sqrt{11}}Y_1^0\right]}_{\text{normalized angular part}} \underbrace{R(r)}_{\text{normalized radial part}}$$

From here we can determine the probabilities:

$$P(l=1, m=-1) = |C_1^{-1}|^2 = \frac{1}{11}$$

$$P(l=1, m=1) = |C_1^{1}|^2 = \frac{1}{11}$$

$$P(l=1, m=0) = |C_1^{0}|^2 = \frac{9}{11}$$

4. a) $1 = \chi^{+}\chi = |A|^2 \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} = |A|^2 \cdot 25$

$A = \frac{1}{5}$ and $\chi = \frac{1}{5}\begin{pmatrix} 3i \\ 4 \end{pmatrix}$

b) $\langle \hat{S}_x \rangle = \frac{\hbar}{2}\langle \hat{\sigma}_x \rangle = \frac{\hbar}{2}\frac{1}{25}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = 0$

$\langle S_y \rangle = \frac{\hbar}{2}\langle \hat{\sigma}_y \rangle = \frac{\hbar}{2}\frac{1}{25}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = \frac{\hbar}{50}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} -4i \\ -3 \end{pmatrix} = -\frac{12}{25}\hbar$

$\langle \hat{S}_z \rangle = \frac{\hbar}{2}\langle \hat{\sigma}_z \rangle = \frac{\hbar}{2}\frac{1}{25}\begin{pmatrix} -3i & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3i \\ 4 \end{pmatrix} = -\frac{7}{50}\hbar$

c) Since we know that $\hat{\sigma}_i^2 = \hat{1}$ $\quad i = x, y, z$

we can say right away that

$\langle \hat{S}_i^2 \rangle = \frac{\hbar^2}{4}\chi^{+}\chi = \frac{\hbar^2}{4}$

$\Delta S_x = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}$ $\qquad \Delta S_y = \sqrt{\frac{\hbar^2}{4} - \frac{144}{625}\hbar^2} = \frac{\hbar}{2}\sqrt{1 - \frac{576}{625}} = \frac{7}{50}\hbar$

$\Delta S_z = \sqrt{\frac{\hbar^2}{4} - \frac{49}{2500}\hbar^2} = \frac{\hbar}{2}\sqrt{1 - \frac{196}{2500}} = \frac{\hbar}{100}\sqrt{2304} = \frac{12}{25}\hbar$

The uncertainty principle: $\Delta S_i \Delta S_j \geq \frac{1}{2}\left|[S_i, S_j]\right| = \frac{\hbar}{2}|\langle S_k \rangle| \quad \begin{matrix} k \neq i \\ k \neq j \end{matrix}$

$\Delta S_x \Delta S_y = \frac{7\hbar^2}{100}$ $\qquad \frac{\hbar}{2}|\langle S_z \rangle| = \frac{7}{100}\hbar^2 \Longrightarrow$ exact equality, consistent

$\Delta S_x \Delta S_z = \frac{12\hbar^2}{50}$ $\qquad \frac{\hbar}{2}|\langle S_y \rangle| = \frac{7}{50}\hbar^2 \Longrightarrow$ consistent

$\Delta S_y \Delta S_z = \frac{84}{1250}\hbar^2$ $\qquad \frac{\hbar}{2}|\langle S_x \rangle| = 0 \Longrightarrow$ consistent