1. $e^{\hat A} = \sum_{k=0}^{\infty} \frac{\hat A^k}{k!}$ $\qquad e^{-\hat A} = \sum_{k=0}^{\infty} \frac{(-\hat A)^k}{k!}$

$$e^{\hat A}\hat B e^{-\hat A} = \left(1 + \hat A + \frac{\hat A^2}{2!} + \frac{\hat A^3}{3!} + \dots\right)\hat B\left(1 - \hat A + \frac{\hat A^2}{2!} - \frac{\hat A^3}{3!} + \dots\right) =$$

$$= \hat B - \hat B\hat A + \frac{\hat B\hat A^2}{2!} - \frac{\hat B\hat A^3}{3!} + \dots + \hat A\hat B - \hat A\hat B\hat A + \frac{\hat A\hat B\hat A^2}{2!} - \frac{\hat A\hat B\hat A^3}{3!} + \dots$$

$$+ \frac{\hat A^2}{2!}\hat B - \frac{\hat A^2}{2!}\hat B\hat A + \frac{\hat A^2}{2!}\hat B\frac{\hat A^2}{2!} - \frac{\hat A^2}{2}\hat B\frac{\hat A^3}{3!} + \dots + \frac{\hat A^3}{3!}\hat B - \frac{\hat A^3}{3!}\hat B\hat A + \frac{\hat A^3}{3!}B\frac{\hat A^2}{2!} - \frac{A^3}{3!}B\frac{A^3}{3!} + \dots$$

$$= B + (\hat A\hat B - \hat B\hat A) + \left(\frac{\hat A^2\hat B}{2!} + \frac{\hat B\hat A^2}{2!} - \hat A\hat B\hat A\right) +$$

$$+ \left(\frac{AB\hat A^2}{2!} - \frac{\hat B\hat A^3}{3!} - \frac{\hat A^2\hat B\hat A}{2!} + \frac{A^3B}{3!}\right) + \dots$$

$$= B + [\hat A, B] + \frac{1}{2!}\left(\hat A^2\hat B - \hat A\hat B\hat A + \hat B\hat A^2 - \hat A\hat B\hat A\right)$$

$$+ \frac{1}{3!}\left(\hat A\hat B\hat A^2 + \hat A\hat B\hat A^2 + \hat A\hat B\hat A^2 - \hat B\hat A^3 - \hat A^2\hat B\hat A - \hat A^2\hat B\hat A - \hat A^2\hat B\hat A + \hat A^3\hat B\right) + \dots =$$

$$= \hat B + [\hat A, B] + \frac{1}{2!}\left(\hat A[\hat A,\hat B] - [\hat A,\hat B]\hat A\right)$$

$$+ \frac{1}{3!}\left(-\hat A[\hat A,\hat B]\hat A + [\hat A,\hat B]\hat A^2 - \hat A[\hat A,\hat B]\hat A + \hat A^2[\hat A,\hat B]\right) + \dots =$$

$$= \hat B + [\hat A,\hat B] + \frac{1}{2!}[\hat A,[\hat A,\hat B]] +$$

$$\frac{1}{3!}\left(-[\hat A,[\hat A,\hat B]]\hat A + \hat A[\hat A,[\hat A,\hat B]]\right) + \dots =$$

$$= \hat B + [\hat A,\hat B] + \frac{1}{2!}[\hat A,[\hat A,\hat B]] + \frac{1}{3!}[\hat A,[\hat A,[\hat A,\hat B]]] + \dots$$

2. If $\langle L^2 \rangle = 6\hbar^2$ then $l(l+1) = 6 \Rightarrow l = 2$

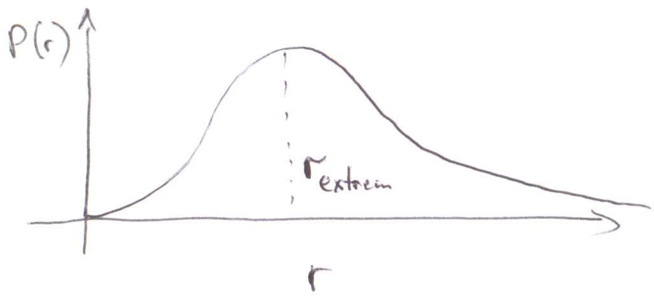
for $l = 2$ possible values of $\langle L_i \rangle$ $(i = x, y, z)$

are $2\hbar, \hbar, 0, -\hbar, -2\hbar$

3. $\psi(\vec{r}) \propto e^{-\frac{r}{a}}$

In spherical coordinates the probability density for r irrespective of direction is

$$P(r) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, |\psi|^2 r^2 \sin\theta = 4\pi r^2 |\psi|^2 \propto r^2 e^{-\frac{2r}{a}}$$



The maximum of $P(r)$ is reached at point where $\frac{\partial P}{\partial r} = 0 \Rightarrow \left(2r - \frac{2}{a} r^2\right) e^{-\frac{2r}{a}} = 0 \Rightarrow$

$\Rightarrow 2r - \frac{2}{a} r^2 = 0 \Rightarrow 1 = \frac{r}{a} \Rightarrow r = a$

So the most probable value or r is $r = a$

4. $$\Psi(\vec{r},t) = \frac{1}{\sqrt{2}}\psi_{211}(\vec{r})\, e^{-\frac{iE_2t}{\hbar}} + \frac{1}{\sqrt{2}}\psi_{21-1}(\vec{r})\, e^{-\frac{iE_2t}{\hbar}} =$$

$$= \frac{e^{-\frac{iE_2t}{\hbar}}}{\sqrt{2}}\frac{1}{\sqrt{24}}\frac{r}{a^{5/2}}\, e^{-\frac{r}{2a}}\left(-\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{i\phi} + \left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\, e^{-i\phi}\right)$$

$$= \frac{e^{-\frac{iE_2t}{\hbar}}}{8\sqrt{2\pi}}\frac{r}{a^{5/2}}\, e^{-\frac{r}{2a}}\sin\theta\,\underbrace{\left(-e^{i\phi}+e^{-i\phi}\right)}_{-2i\sin\phi} =$$

$$= -\frac{i}{4\sqrt{2\pi}\,a^{5/2}}\, e^{-\frac{iE_2t}{\hbar}}\, r e^{-\frac{r}{2a}}\sin\theta\sin\phi$$

$$V = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$$

$$\langle V\rangle = \int \Psi^*(\vec{r},t)\, V\, \Psi(\vec{r},t)\, d\vec{r} =$$

$$= \frac{1}{16\cdot 2\pi\, a^5}\left(-\frac{e^2}{4\pi\varepsilon_0}\right)\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty}\left(r e^{-\frac{r}{2a}}\sin\theta\sin\phi\right)^2 \frac{1}{r}\, r^2 \sin\theta\, dr =$$

$$= -\frac{e^2}{128\pi^2 a^5\varepsilon_0}\cdot\underbrace{\int_0^{2\pi}\sin^2\phi\, d\phi}_{\pi}\;\underbrace{\int_0^{\pi}\sin^3\theta\, d\theta}_{\frac{4}{3}}\;\underbrace{\int_0^{\infty} r^3 e^{-\frac{r}{a}}\, dr}_{6a^4} =$$

$$= -\frac{e^2}{16\pi a\varepsilon_0}$$

5. There are no bound states with $\ell>0$ since the "effective" potential, $V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2} = \frac{\hbar^2}{2m}\frac{(\ell(\ell+1)-2)}{r^2} > 0$ is purely repulsive for $\ell>0$