① For a time-independent harmonic oscillator potential we have the following time-dependent wave function:

$$|\psi(t)\rangle = \frac{1}{\sqrt{3}}\left(e^{-\frac{iE_1 t}{\hbar}}|1\rangle + e^{-\frac{iE_2 t}{\hbar}}|2\rangle + e^{-\frac{iE_3 t}{\hbar}}|3\rangle\right)$$

where $E_n = \hbar\omega(n+\frac{1}{2})$

we can rewrite $\psi(t)$ as follows

$$|\psi(t)\rangle = \frac{e^{-\frac{3}{2}i\omega t}}{\sqrt{3}}\left(|1\rangle + e^{-i\omega t}|2\rangle + e^{-2i\omega t}|3\rangle\right)$$

For the expectation value of the position we have

$$\langle\psi(t)|x|\psi(t)\rangle = \frac{1}{3}\left(\langle 1| + \langle 2|e^{i\omega t} + \langle 3|e^{2i\omega t}\right)x\left(|1\rangle + e^{-i\omega t}|2\rangle + e^{-2i\omega t}|3\rangle\right)$$

Using the values of the matrix elements $\langle n|x|k\rangle$ (from the formula sheet)

$$\langle n|x|k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$$

we obtain:

$$\langle\psi(t)|x|\psi(t)\rangle = \frac{1}{3}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t}\sqrt{2} + e^{i\omega t}\sqrt{2} + e^{-i\omega t}\sqrt{3} + e^{i\omega t}\sqrt{3}\right) =$$

$$= \frac{\sqrt{2}+\sqrt{3}}{3}\sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$$

The expectation value of the energy is

$$E(t) = \left(\frac{1}{\sqrt{3}}\right)^2 E_1 + \left(\frac{1}{\sqrt{3}}\right)^2 E_2 + \left(\frac{1}{\sqrt{3}}\right)^2 E_3 = \frac{1}{3}\left(\frac{3}{2}\hbar\omega + \frac{5}{2}\hbar\omega + \frac{7}{2}\hbar\omega\right) =$$

$$= \frac{5}{2}\hbar\omega$$

② a) Not Hermitian:

$$\left(|1\rangle\langle 2| + i|2\rangle\langle 1|\right)^{\dagger} = |2\rangle\langle 1| - i|1\rangle\langle 2| \neq |1\rangle\langle 2| + i|2\rangle\langle 1|$$

b) Hermitian:

$$\left(|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 4| + |4\rangle\langle 3|\right)^{\dagger} = |1\rangle\langle 1| + |2\rangle\langle 2| + |4\rangle\langle 3| + |3\rangle\langle 4|$$

c) Hermitian

$$(a|1\rangle + b^*|2\rangle)^{\dagger}(b|1\rangle - a^*|2\rangle)|3\rangle\langle 2| + |4\rangle\langle 4| = |4\rangle\langle 4|$$

d) Hermitian

$$\left(|1\rangle\langle 1| + i|2\rangle\langle 1| - i|1\rangle\langle 2| + |2\rangle\langle 2|\right)^{\dagger} = |1\rangle\langle 1| - i|1\rangle\langle 2| + i|2\rangle\langle 1| + |2\rangle\langle 2|$$

e) No. For a projection operator $A^2 = A$. In our case we have $A = \frac{1}{\sqrt{2}}(1 + |\psi\rangle\langle\psi|)$ and

$$A^2 = \frac{1}{2}(\hat{1} + |\psi\rangle\langle\psi|)^2 = \frac{1}{2}\left(\hat{1} + |\psi\rangle\langle\psi| + |\psi\rangle\langle\psi| + |\psi\rangle\langle\psi|\psi\rangle\langle\psi|\right) =$$

$$= \frac{1}{2}(\hat{1} + 3|\psi\rangle\langle\psi|) \neq A$$

So, no, A is not a projection operator.

f) $$\exp(i\pi\hat{I})f(x) = \left(1 + i\pi\hat{I} - \frac{\pi^2\hat{I}^2}{2!} - \frac{i\pi^3\hat{I}^3}{3!} + \frac{\pi^4\hat{I}^4}{4!} + \dots\right)f(x) =$$

$$= \left(1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \dots\right)f(x) + i\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \dots\right)f(-x) =$$

$$= \cos\pi\, f(x) + i\sin\pi\, f(-x) = -f(x)$$

So

$$\exp(i\pi\hat{I}) = -\hat{1}$$

g) We know that $\left(\frac{d}{dx}\right)^{\dagger} = -\frac{d}{dx}$ (recall that $p^{\dagger} = p$ and $p = -i\hbar\frac{d}{dx}$)

Therefore $\left(\frac{d}{dx}\right)^{2\dagger} = \frac{d^2}{dx^2}$ $\quad\left(\frac{d}{dx}\right)^{3\dagger} = -\frac{d^3}{dx^3}$ $\quad\left(\frac{d}{dx}\right)^{4\dagger} = \frac{d^4}{dx^4}$

and $\left(\frac{d^5}{dx^5}\right)^{\dagger} = -\frac{d^5}{dx^5}$

h) $\hat{S}_y = \frac{\hbar}{2}\hat{\sigma}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

The eigenvalues and eigenstates of $\hat{S}_y$ are:

$\frac{\hbar}{2}$, $|\frac{1}{2}\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $-\frac{\hbar}{2}$, $|-\frac{1}{2}\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

The projector on the state $|\frac{1}{2}\rangle_y$ is:

$$\underline{P} = |\tfrac{1}{2}\rangle_y \langle\tfrac{1}{2}|_y = \frac{1}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}\begin{pmatrix} 1 & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

③ Let the radial wave function of the particle be $R(r) = \frac{u(r)}{r}$. Function $u(r)$ satisfies the following 1D differential equation:

$$\frac{d^2u}{dr^2} + \left[\frac{2m}{\hbar^2}\left(E - V(r)\right) - \frac{l(l+1)}{r^2}\right]u = 0 \qquad a \leq r \leq b$$

For the ground state we know that $l = 0$. So the angular part of the wave function is a constant, while the above equation for $u(r)$ is further simplified to:

$$u'' + k^2 u = 0 \quad , \quad \text{where} \quad k = \frac{\sqrt{2mE}}{\hbar}$$

The boundary conditions are:

$$u(r=a) = u(r=b) = 0$$

The general solution is

$$u(r) = A \sin(kr + \phi)$$

The condition $u(a) = 0$ gives that $\phi = -ka$

While the condition $u(b) = 0$ gives the quantization of energy levels:

$$k(b-a) = n\pi \qquad n = 1, 2, 3, \ldots$$

Then we can write the energy of the ground state:

$$E_1 = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2m(b-a)^2}$$

Constant A is found from the normalization: $\int_a^b R^2(r) r^2 dr = 1$

$$1 = A^2 \int_a^b \sin^2(k(r-a))\, dr = A^2\left[\frac{b-a}{2}\right] \Rightarrow A = \sqrt{\frac{2}{b-a}}$$

(4) First, we need to find the explicit form of 3×3 matrices S_x, S_y, and S_z. The procedure is described in detail in the lecture on the matrix representation of the angular momentum operator. The result is:

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Using these we can compute the matrix of the Hamiltonian:

$$H = \alpha S_z^2 + \beta(S_x^2 - S_y^2) = \hbar^2\begin{pmatrix} \alpha & 0 & \beta \\ 0 & 0 & 0 \\ \beta & 0 & \alpha \end{pmatrix}$$

Laslty, we can find the eigenvalues and eigenvectors of this matrix

$$E_1 = 0 \qquad \psi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$E_2 = \alpha - \beta \qquad \psi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$E_3 = \alpha + \beta \qquad \psi_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(5) See lecture 24 (Chains of 1D wells) for details. The resulting matrix of the Hamiltonian in the case of three wells is:

$$H = \begin{pmatrix} \varepsilon & \beta & 0 \\ \beta & \varepsilon & \beta \\ 0 & \beta & \varepsilon \end{pmatrix}$$

where $\varepsilon = \langle 1|H|1\rangle = \langle 2|H|2\rangle = \langle 3|H|3\rangle$

and $\beta = \langle 1|H|2\rangle = \langle 2|H|3\rangle = $
$= \langle 2|H|1\rangle = \langle 3|H|2\rangle$

we assume that $\langle 1|H|3\rangle$ is negligible. $|1\rangle$, $|2\rangle$, and $|3\rangle$ here stand for $|g(x+a)\rangle$, $|g(x)\rangle$, and $|g(x-a)\rangle$ respectively.

The energies of the above Hamiltonian are

$$E_1 = \varepsilon - \sqrt{2}\beta \qquad E_2 = \varepsilon \qquad E_3 = \varepsilon + \sqrt{2}\beta$$