Dirac delta function (review)

The Dirac delta function is a generalized function that can be defined as a limiting case of a function that becomes infinitely narrow (i.e. highly localized around a single point) and at the same time infinitely tall. Importantly, the transition to the limit must be done in such a way that the area under the curve (the integral) remains equal to unity. In other words, the delta function can be thought as an "impule" or "spike" with a finite area under the curve that describes it.

From the point of view of the "classical" calculus, the delta function is not a well defined function. As a mathematical object it only makes sense when it appears inside an integral. Formally we can say that

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

The concept of the delta function is convenient when it is necessary to describe distributions that are extremely localized (e.g. the charge distribution of a point charge)

Since the delta function is peaked at a single point ($x=0$) only, it is easy to see that for any reasonably nice function $f(x)$ we have

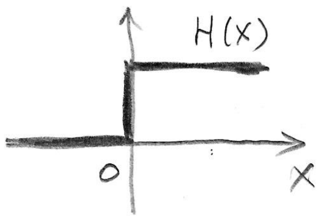
$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0) \quad \text{or} \quad \int_{-\infty}^{+\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

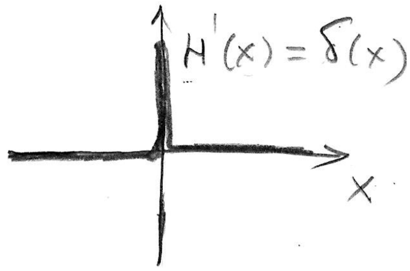
In fact,

$$\int_{x_0-\epsilon}^{x_0+\epsilon} f(x)\,\delta(x-x_0)\,dx = f(x_0) \quad \text{for any } \epsilon > 0$$

The delta function can also be thought of as the derivative of the Heaviside step function

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$



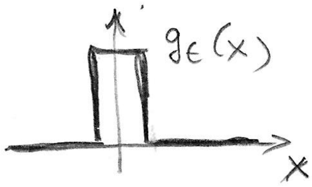


The delta function is a continuous analogue of the Kronecker delta symbol: $\delta(x-y) \Leftrightarrow \delta_{ij}$

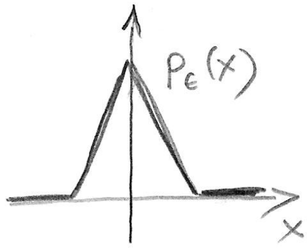
Here are a few examples of limiting cases that converge to the delta function:

$$\lim_{\epsilon \to 0^+} \frac{1}{\sqrt{\pi}\,\epsilon} e^{-\frac{x^2}{\epsilon^2}} \qquad \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\sin\left(\frac{x}{\epsilon}\right)}{x}$$

$$\lim g_\epsilon(x) \qquad g_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon}, & -\epsilon \leq x \leq \epsilon \\ 0, & \text{otherwise} \end{cases}$$



$$\lim p_\epsilon(x) \qquad p_\epsilon(x) = \begin{cases} \frac{1}{\epsilon^2}(x+\epsilon), & -\epsilon \leq x < 0 \\ \frac{1}{\epsilon^2}(-x+\epsilon), & 0 < x \leq \epsilon \\ 0, & \text{otherwise} \end{cases}$$



Let us list the most important properties of the delta function:

1) $\delta(-x) = \delta(x)$

2) $\delta(ax) = \frac{1}{|a|}\delta(x) \quad a = \text{const}$

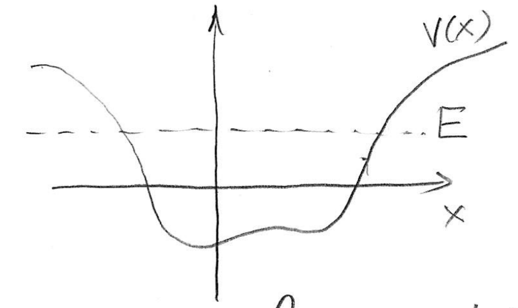
3) $x\,\delta(x) = 0$ 4) $x\,\delta'(x) = -\delta(x)$ 5) $\delta'(-x) = -\delta'(x)$

6) $\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$ where $f'(x) = \frac{df}{dx}$ and x_i's are simple roots of $f(x)=0$

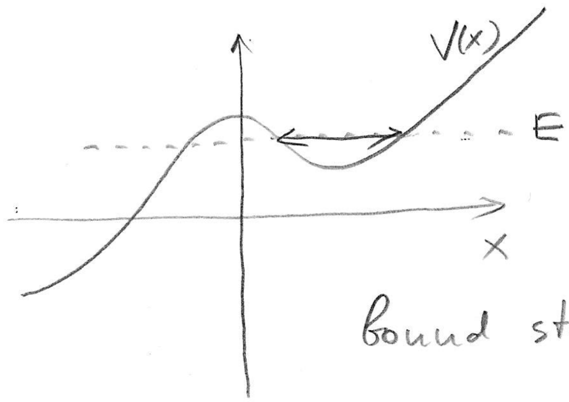
7) $\delta(x-y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(x-y)t}\,dt$

General properties of 1D motion in quantum mechanics. Bound and scattering states.

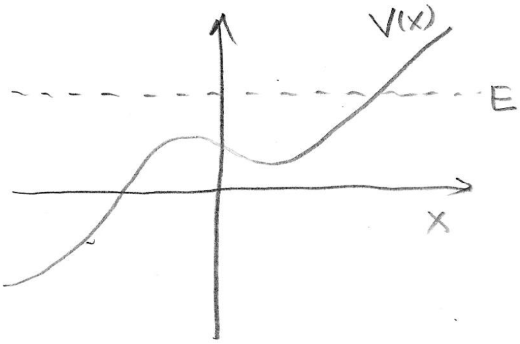
In classical mechanics, depending on the shape of the potential, there are three distinct possibilities for the motion in 1D



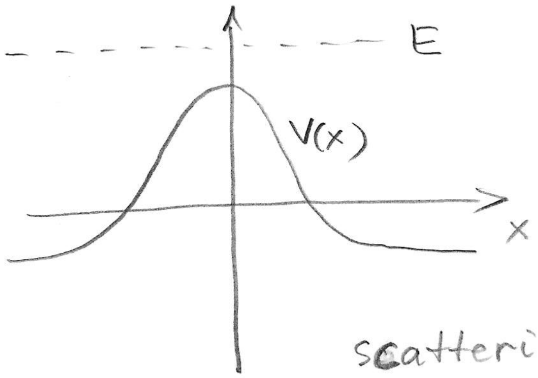
Bound state



Bound state



Scattering (semi-infinite motion)



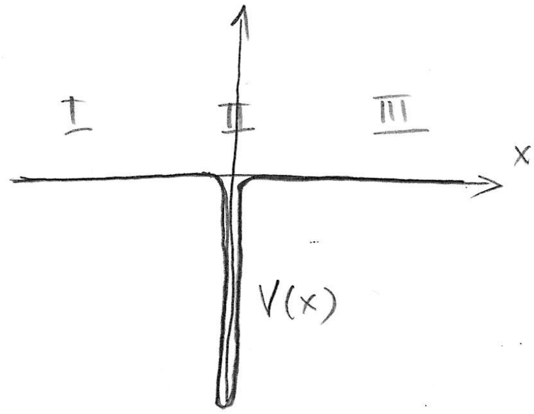
scattering (infinite motion)

Similarly, in quantum mechanics the motion may take different forms. The solutions of the Schrödinger equation can correspond either to bound states (discrete energy spectrum) or scattering states (continuous spectrum).

Attractive delta function potential

Let us consider the potential $V(x) = -\alpha\,\delta(x)$, where α is some positive constant. The Schrödinger equation has the following form then:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi = E\psi$$



In region I $(x<0)$ it becomes

$$\frac{d^2\psi}{dx^2} = k^2\psi \qquad k = \frac{\sqrt{-2mE}}{\hbar}$$

where we assume that $E<0$ (we first invistigate the possibility of bound states)

The general solution is $\psi = Ae^{-kx} + Be^{kx}$. However, the square integrability requires that $A=0$, thus

$$\psi = Be^{kx} \qquad (x<0)$$

In region III $(x>0)$ we obtain, in a similar way, that

$$\psi = Fe^{-kx}$$

Now we have to "match" these two parts of the solution. ψ is a continuous function. ψ' may have discontinuities at the points where the potential is singular. In our case $B=F$ and we can write that

$$\psi = Be^{-k|x|}$$

B can be determined from the normalization condition:

$$\int_{-\infty}^{+\infty} |\psi|^2 dx = 1 \quad \Rightarrow \quad B = \sqrt{k}$$

But what about κ? Is it quantized?

Let us integrate the Schrödinger equation around the point $x=0$ from $-\epsilon$ to $+\epsilon$, where $\epsilon \to 0$

$$\underbrace{-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}dx}_{-\frac{\hbar^2}{2m}\left[\psi'(+\epsilon)-\psi'(-\epsilon)\right]} + \underbrace{\int_{-\epsilon}^{+\epsilon}V(x)\,\psi(x)\,dx}_{-\alpha\psi(0)} = \underbrace{E\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx}_{0\text{, because } \epsilon\to 0 \text{ and } \psi \text{ is a "nice" function}}$$

$$\psi'(-\epsilon) = \sqrt{\kappa}\,\kappa\, e^{\kappa\epsilon} = \kappa^{3/2} \qquad \epsilon \to 0$$

$$\psi'(\epsilon) = \sqrt{\kappa}\,\kappa\, e^{-\kappa\epsilon} = -\kappa^{3/2} \qquad \epsilon \to 0$$

$$\psi(0) = \sqrt{\kappa}$$

With that we obtain the following relation:

$$-\frac{\hbar^2}{2m}\left[-\kappa-\kappa\right] = \alpha \implies \frac{\hbar^2\kappa}{m} = \alpha \implies \kappa = \frac{\alpha m}{\hbar^2}$$

What we see is that there is only one value of κ allowed (for negative energies); The corresponding energy is

$$E = -\frac{m\alpha^2}{2\hbar^2}$$

Interestingly, this number of discrete energy levels (1) is independent of the strength of $V(x)$, i.e. independent of the magnitude of α.

Now let us consider the case when $E>0$ (but the potential is still the same with $\alpha>0$)

$$\frac{d^2\psi}{dx^2} = -k^2\psi \qquad k = \frac{\sqrt{2mE}}{\hbar} \qquad \psi = Ae^{ikx} + Be^{-ikx}, \; x<0$$

$$\psi = Fe^{ikx} + Ge^{-ikx} \qquad x>0$$

The continuity of the wave function at $x=0$ gives.

$$F+G = A+B$$

At the same time

$$\psi' = \begin{cases} ik\left(Fe^{ikx} - Ge^{-ikx}\right), & x>0 \\ ik\left(Ae^{ikx} - Be^{-ikx}\right), & x<0 \end{cases}$$

which gives

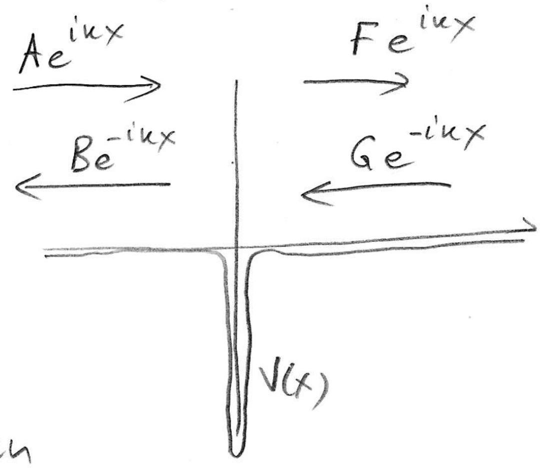
$$\psi'(0^+) = ik(F-G) \qquad \psi'(0^-) = ik(A-B)$$

Integrating the Schrödinger equation (in the same way as we did previosly) yields

$$ik\left(F-G-A+B\right) = -\frac{2m\alpha}{\hbar^2}(A+B), \quad \text{or}$$

$$F-G = A(1+2i\beta) - B(1-2i\beta) \quad \text{where } \beta = \frac{m\alpha}{\hbar^2 k}.$$

e^{ikx} represents a wave propagating in the positive direction of X (recall $e^{ikx - i\frac{Et}{\hbar}} = e^{ik\underbrace{(x - \frac{\hbar k t}{2m})}_{x-vt}}$). Thus, A is the amplitude of the wave that comes from the left, B is the amplitude of the wave traveling to the right, and G is the amplitude of the wave that comes from the right.



In the case of scattering from the left $G=0$ and

$$B = \frac{i\beta}{1-i\beta}A \qquad F = \frac{1}{1-i\beta}A$$

The relative probability that an incident particle is reflected back is

$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2} \quad \text{reflection coefficient}$$

$$T \equiv \frac{|F|^2}{|A|^2} = \frac{1}{1+\beta^2} \quad \text{transmission coefficient}$$

If we now change the sign of α, we no longer get any bound states $(E<0)$. At the same time, R and T remain unchanged, i.e the reflection and transmission coefficients for the delta function potential are independent of the sign of α.