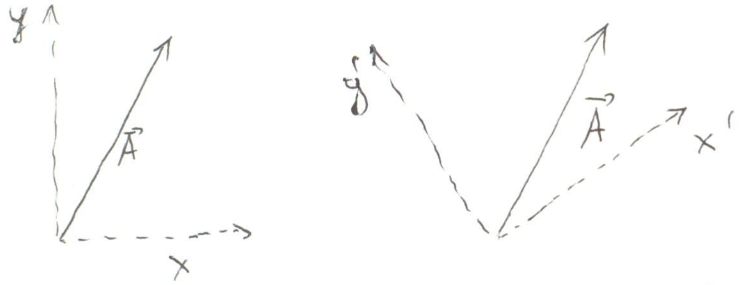
Dirac notation

Notation – the way we choose to organize our symbology to represent something – can play a powerful role in helping us think about complex manipulations with mathematical objects. In Maxwell's day, the equations for electric and magnetic fields were written out component by component, so his equations took up a full page of text. Looking at those equations, it is clear that there is regularity to the equations that should allow for some compression. When Gibbs introduced his vector notation, Maxwell's equations could be collapsed into 4 lines. Furthermore, they had the advantage that they did not depend on the choice of coordinate system. You could use the same equations, manipulate them as you wished, and then introduce a particular choice of coordinate (e.g. a particular orientation of rectangular coordinates or a convenient set of curvilinear coordinates) after you were done.

The motivation for Dirac notation (also known as bra-ket notation) stems from the same two reasons: to make writing thing more compact and meaningful and to abstract from a particular choice of representation (basis). It was introduced by P. Dirac in 1939.

Bra-ket notation is extremely widespread in quantum physics. Almost every phenomenon described using quantum mechanics is usually described with the help of bra-ket notation.

Consider an ordinary vector $\vec{A}$ is two dimensions. No matter which set of axes we choose



$$\vec{A} = (A_x, A_y)$$

$$\vec{A}' = (A'_x, A'_y)$$

it remains the same vector. The same is true for the state of a system in quantum mechanics. It is represented by some vector $|S(t)\rangle$. We can express it with respect to any basis. The wave function $\Psi(x,t)$ which we have delt most often up to now, is actually the coefficient in the expansion of $|S\rangle$ in the basis of position eigenfunctions:

$$\Psi(x,t) = \langle x | S(t) \rangle$$

Here $|x\rangle$ stands for the eigenfunction of $\hat{x}$ with eigenvalue x.

Similarly the momentum space wavefunction is the expansion of $|S\rangle$ in the basis of momentum eigenfunctions:

$$\Phi(p,t) = \langle p | S(t) \rangle$$

We could also expand $|S\rangle$ in the basis of energy eigenfunctions:

$$C_n(t) = \langle n | S(t) \rangle$$

Where $|n\rangle$ stands for the n-th eigenfunctions of the Hamiltonian.

In any event, it is all the same state described in three different ways:

$$\Psi(x,t) = \int \Psi(y,t)\,\delta(x-y)\,dy = \int \Phi(p,t)\,\frac{e^{\frac{ipx}{\hbar}}}{\sqrt{2\pi\hbar}}\,dp =$$

$$= \sum_n c_n(t)\,e^{-\frac{iE_nt}{\hbar}}\,\psi_n(x)$$

Vectors represented with respect to a particular basis $\{|e_n\rangle\}$ as

$$|\alpha\rangle = \sum_n a_n |e_n\rangle \qquad \text{where } a_n = \langle e_n|\alpha\rangle$$

$$|\beta\rangle = \sum_n b_n |e_n\rangle \qquad \text{where } b_n = \langle e_n|\beta\rangle$$

Similarly operators (matrices) are represented with respect to a particular basis by their matrix elements

$$\langle e_m|\hat{Q}|e_n\rangle = Q_{mn}$$

$$|\beta\rangle = Q|\alpha\rangle \iff \sum_n b_n|e_n\rangle = \sum Q\,a_n|e_n\rangle$$

$$\sum_n b_n\langle e_m|e_n\rangle = \sum a_n\langle e_m|Q|e_n\rangle$$

$$b_m = \sum_n Q_{mn}\,a_n$$

Dirac proposed to denote the inner product as

$$\underbrace{\langle\alpha|}_{\text{bra}}\underbrace{\beta\rangle}_{\text{ket}}$$

The bra-vector is an analogue of a transposed and complex conjugate vector (i.e. row-vector) in linear algebra.

Whenever we encounter a product $\langle \alpha |$ and $| \beta \rangle$ we assume the inner product. (dot product if we deal with the usual vectors in linear algebra)

A combination of $|\alpha\rangle$ and $\langle\beta|$ yields the outer-product, which is an analogue of the Kronecker product in linear algebra)

$$|\alpha\rangle\langle\beta| \iff \vec{a} \otimes \vec{b}^* = \begin{pmatrix} a_1 b_1^* & a_1 b_2^* & a_1 b_3^* & \cdots \\ a_2 b_1^* & a_2 b_2^* & a_2 b_3^* & \cdots \\ a_3 b_1^* & a_3 b_2^* & a_3 b_3^* & \cdots \\ \vdots & & & \end{pmatrix}$$

Dirac notation allows to formalize many manipulations without using a particular representation (basis)

For example the projection operator $\hat{P} \equiv |\alpha\rangle\langle\alpha|$ separates out the portion of vector that lies along $|\alpha\rangle$:

$$\underline{P} |\beta\rangle = |\alpha\rangle \underbrace{\langle\alpha|\beta\rangle}_{\text{component of } |\beta\rangle \text{ along } |\alpha\rangle} = \langle\alpha|\beta\rangle |\alpha\rangle$$

For a complete orthonormal basis

$$\sum_n |e_n\rangle\langle e_n| = 1$$

Indeed any vector $|\alpha\rangle$ can be expanded in terms of $|e_n\rangle$: $|\alpha\rangle = \sum_m a_m |e_m\rangle$

$$\sum_n |e_n\rangle\langle e_n| \cdot \sum_m a_m |e_m\rangle = \sum_{n,m} a_m |e_n\rangle \underbrace{\langle e_n|e_m\rangle}_{\delta_{nm}} = \sum_n a_n |e_n\rangle$$

Interestingly, the Dirac notation can be used in the case of both discrete and continuous basis (or a combination of both). We just need to replace the summation with integration:

$$\sum_{n} |e_n\rangle\langle e_n| \Longrightarrow \int |e_n\rangle\langle e_n|\, dz \quad \text{with } \langle e_n|e_{n'}\rangle = \delta(n-n')$$

In fact in many cases when physicist write a summation they assume integration. This situation is similar to the one when we deal with the Fourier series, which becomes the Fourier integral transform when the range approaches infinity.

We can represent any operator in terms of its spectral decomposition:

$$Q|e_n\rangle = q_n|e_n\rangle$$

$$Q = \sum_{n} |e_n\rangle q_n \langle e_n|$$

It is easy to verify that when Q acts on any vector $|\alpha\rangle$ we get

$$|\alpha\rangle = \sum_{m} a_n|e_m\rangle$$

$$Q|\alpha\rangle = \sum_{nm} q_n|e_n\rangle \underbrace{\langle e_n|a_n|e_m\rangle}_{a_n\delta_{nm}} = \sum_{n} q_n a_n|e_n\rangle$$