Orbital angular momentum

To treat problems with rotational symmetry, it is useful to introduce the orbital angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$:

$$\hat{\vec{L}} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \hat{x} & \hat{y} & \hat{z} \\ \hat{p}_x & \hat{p}_y & \hat{p}_z \end{vmatrix}$$

or in components

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \qquad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \qquad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

$\hat{\vec{L}}$ is a Hermitian operator, i.e. $\hat{\vec{L}}^+ = \hat{\vec{L}}$, which can be easily verified. For example

$$\hat{L}_z^+ = \hat{p}_y^+ \hat{x}^+ - \hat{p}_x^+ \hat{y}^+ = \hat{p}_y \hat{x} - \hat{p}_x \hat{y} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \hat{L}_z$$

In the above check we used the fact that

$$[\hat{r}_i, \hat{p}_j] = i\hbar \delta_{ij} \qquad i,j = 1,2,3$$

$$\hat{\vec{r}} = \begin{pmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad \hat{\vec{p}} = \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \end{pmatrix} = \begin{pmatrix} \hat{p}_x \\ \hat{p}_y \\ \hat{p}_z \end{pmatrix}$$

For simplicity we will no longer use hats to denote operators unless there is ambiguity. In most cases it is obvious when we deal with operators.

An essential property of the angular momentum operator is that its components do not commute with one another:

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x] + [zp_y, xp_z] =$$

$$= y[p_z, z]p_x + x[z, p_z]p_y = i\hbar(xp_y - yp_x) = i\hbar L_z$$

Similarly, $[L_y, L_z] = i\hbar L_x$ $\quad [L_z, L_x] = i\hbar L_y$

This can also be written as

$$[L_i, L_j] = i\hbar \varepsilon_{ijk} L_k$$

where ε_{ijk} is the antisymmetric tensor

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$$

$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj}$$

ε_{ijk} is known as the Levi-Civita symbol

Another compact way to express the commutation relations between L_i and L_j is

$$\vec{L} \times \vec{L} = i\hbar \vec{L}$$

Note that the vector (cross) product would be zero if $\vec{L}$ were an ordinary ~~vectors~~ vector (not operator)

Since L_x, L_y, and L_z do not commute, they correspond to incompatible observables. The above commutation relations imply the uncertainty relations

$$\Delta L_x \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle| \qquad \left(\text{recall } \Delta A \Delta B \geq \frac{1}{2} \langle [A,B] \rangle\right)$$

where Δ, as usual, denotes the standard deviation

Therefore, it is impossible to know simultaneously two different components of $\vec{L}$ (unless $\langle \vec{L} \rangle = 0$)

However, one may notice that each of the components of $\vec{L}$ commutes with $\vec{L}^2$:

$$[\vec{L}^2, L_i] = 0 \qquad \vec{L}^2 = L_x^2 + L_y^2 + L_z^2$$

For example

$$[L^2, L_x] = \underbrace{[L_x^2, L_x]}_{0} + [L_y^2, L_x] + [L_z^2, L_x] =$$

$$= L_y \underbrace{[L_y, L_x]}_{-i\hbar L_z} + \underbrace{[L_y, L_x]}_{-i\hbar L_z} L_y + L_z \underbrace{[L_z, L_x]}_{i\hbar L_y} + \underbrace{[L_z, L_x]}_{i\hbar L_y} L_z$$

$$= 0$$

Therefore, L^2 is compatible with each component of $\vec{L}$. We can find simultaneous eigenstates of L^2 and (say) L_z:

$$L^2 f = \lambda f \qquad L_z f = \mu f$$

The ladder operator method for angular momentum

Define two new operators:

$$L_+ = L_x + iL_y \qquad L_- = L_x - iL_y = L_+^\dagger$$

Clearly $[\vec{L}^2, L_\pm] = 0$

$$[L_z, L_+] = [L_z, L_x + iL_y] = i\hbar L_y + i(-i\hbar L_x) = \hbar(L_x + iL_y) = \hbar L_+$$

$$[L_z, L_-] = i\hbar L_y - \hbar L_x = -\hbar L_-$$

$$[L_+, L_-] = [L_x + iL_y, L_x - iL_y] = -i[L_x, L_y] + i[L_y, L_x] = 2\hbar L_z$$

Furthermore, $L_+L_- = (L_x + iL_y)(L_x - iL_y) =$

$= L_x^2 + L_y^2 - i[L_x, L_y] = L_x^2 + L_y^2 + \hbar L_z$

$L_-L_+ = L_x^2 + L_y^2 + i[L_x, L_y] = L_x^2 + L_y^2 - \hbar L_z$

Thus we can write L^2 as

$$L^2 = \frac{1}{2}(L_+L_- + L_-L_+) + L_z^2$$

From the commutator $[L_z, L_+] = \hbar L_+$ we have: $L_z L_+ = L_+(L_z + \hbar)$

When acting on f (eigenfunction of L^2 and L_z) it yields

$$L_z L_+ f_{\lambda\mu} = L_+(L_z + \hbar) f_{\lambda\mu} = (\mu + \hbar)\ (L_+ f_{\lambda\mu})$$

$\Rightarrow$ $L_+ f_{\lambda\mu}$ is an eigenstate of L_z with eigenvalue $(\mu + \hbar)$.

Similarly

$$L_z L_- f_{\lambda\mu} = (\mu - \hbar) L_- f_{\lambda\mu}$$

L_+ and L_- are known as the raising and lowering operators

For a given value of λ we obtain a ladder of states, whose eigenvalues are evenly separated by $\hbar$. To ascend the ladder we apply L_+, to descend we apply L_-. This process, however, cannot go forever. Eventually we will reach a state for which the z-component exceeds the total (λ)

Hence the must be some maximum such that

$$L_+ f_{top} = 0$$

Now, let $\hbar l$ be this maximum L_z eigenvalue:

$$L_z f_{top} = \hbar l f_{top} \qquad L^2 f_{top} = \lambda f_{top}$$

Using the fact that $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$

we obtain that

$$L^2 f_{top} = (L_- L_+ + L_z^2 + \hbar L_z) f_{top} = (0 + \hbar^2 l^2 + \hbar^2 l) f_{top} = \hbar^2 l (l+1) f_{top}$$

Thus,

$$\lambda = \hbar^2 l(l+1)$$

This defines the eigenvalue of L^2 in terms of the maximum eigenvalue of L_z

Similarly, there is a bottom limit f_{bot} such that $L_- f_{bot} = 0$:

$$L_z f_{bot} = \hbar l' f_{bot} \qquad L^2 f_{bot} = \lambda f_{bot}$$

$$L^2 f_{bot} = (L_+ L_- + L_z^2 - \hbar L_z) f_{bot} = (0 + \hbar^2 l'^2 - \hbar^2 l') f_{bot} =$$
$$= \hbar^2 l' (l'-1) f_{bot}$$

and

$$\lambda = \hbar^2 l'(l'-1)$$

We now have $l(l+1) = l'(l-1)$

Since l' cannot be equal to $l+1$, the only solution is

$$l' = -l$$

The eigenvalues of L_z are $\mu = m\hbar$, where
~~m~~ runs from $-\ell$ to $+\ell$ in N integer steps

$$\ell = -\ell + N$$

Hence $\ell = N/2$ and ℓ must be either an integer
or half-integer.

The eigenfunctions are characterized by the numbers
ℓ and m :

$$L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m \qquad L_z f_\ell^m = \hbar m f_\ell^m$$

where $\ell = 0, 1/2, 1, 3/2, \ldots \qquad m = -\ell, -\ell+1, \ldots \ell-1, \ell$

There are $2\ell+1$ different m's for each given ℓ.