Addition of angular momenta in quantum mechanics

In this lecture we will examine the relation between the angular momentum of a total system and that of its constituents. This problem is important in atomic, molecular, and nuclear physics, where one encounters systems of few particles (e.g. electrons, nucleons, etc.)

Let us consider two particles (systems) rotating about a common origin. If the angular momentum of the first particle is $\vec{L}_1$ and that of the second particle is $\vec{L}_2$, the magnitude and z component of the total angular momentum of the composite system is

$$\hat{L}^2 = (\hat{\vec{L}}_1 + \hat{\vec{L}}_2)^2 = \hat{L}_1^2 + \hat{L}_2^2 + 2\hat{\vec{L}}_1\hat{\vec{L}}_2$$

$$\hat{L}_z = \hat{L}_{1z} + \hat{L}_{2z}$$

If we assume the system in a state with definite values of L_{1z}, L_{2z} (e.g. $|m_1 m_2\rangle$) the how much further this state be resolved? Since there are only two good quantum numbers associated with each electron (i.e. m_1 l_1) one suspects that the composite system will be characterized by 4 quantum numbers. The state then can be further resolved to $|l_1 l_2 m_1 m_2\rangle$. If the system is in the state $|l_1 l_2 m_1 m_2\rangle$ before measurement we do not know whether it will be in that state after the measurement of L^2. This is because

$$[\hat{L}_{1z}, \hat{L}^2] = [\hat{L}_{1z}, \hat{L}_1^2 + \hat{L}_2^2 + 2\hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2] = 2[\hat{L}_{1z}, \hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2] =$$

$$= 2[\hat{L}_{1z}, L_{1x}L_{2x} + L_{1y}L_{2y} + L_{1z}L_{2z}] =$$

$$= 2i\hbar(\hat{L}_{1y}\hat{L}_{2x} - \hat{L}_{1x}\hat{L}_{2y})$$

$l_1\, l_2\, m_1\, m_2$ are good quantum numbers (i.e. these values may be simultaneously specified) only when the corresponding operators mutually commute. The fact that no other commuting operators can be attached (restricting the consideration to those operators that act on the same coordinates/variables) indicates that this this set is a complete set of commuting operators.

Indeed

$$[\hat{L}_{1z}, \hat{L}_{2z}] = [\hat{L}_{1z}, \hat{L}_1^2] = [\hat{L}_{1z}, \hat{L}_2^2] = [\hat{L}_{2z}, \hat{L}_1^2] =$$
$$= [\hat{L}_{2z}, \hat{L}_2^2] = [\hat{L}_1^2, \hat{L}_2^2] = 0$$

Suppose now that we measure $\hat{L}^2$ and L_z and obtain l and m. Can this state be further resolved? The answer is yes because we can measure L_1^2 and L_2^2 without destroying the eigenstates of L^2 and L_z already established. After measurement the system is left in the state $|l m l_1 l_2\rangle$. To show that l, m, l_1, l_2 are good quantum numbers we must demonstrate that the set $\hat{L}_1^2, \hat{L}_2^2, \hat{L}^2, \hat{L}_z$ is a set of mutually commuting operators. Indeed,

$$[\hat{L}_1^2, \hat{L}^2] = [\hat{L}_1^2, \hat{L}_1^2 + \hat{L}_2^2 + 2\hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2] = 2[\hat{L}_1^2, \hat{\vec{L}}_1 \cdot \hat{\vec{L}}_2] =$$
$$= 2[\hat{L}_1^2, \hat{\vec{L}}_1] \cdot \hat{\vec{L}}_2 = 0$$

and

$$[\hat{L}_1^2, \hat{L}_z] = [\hat{L}_1^2, \hat{L}_{1z} + \hat{L}_{2z}] = [\hat{L}_1^2, L_{1z}] = 0$$

The other mutual commutators are obvious and are also equal to zero

Thus we find that the system can be characterized by either of two sets of good quantum numbers

They correspond to states $|l_1 l_1 m_1 m_2\rangle$ and $|lm\, l_1 l_2\rangle$ respectively. The representation where where L^2 and L_z are specified is called the coupled representation. the representation where $L_1^2, L_{1z}, L_2^2, L_{2z}$ are specified is called uncoupled representation

The eigenstates in either representation are constructed from products of the eigenstates $|l_1 m_1\rangle$ and $|l_2 m_2\rangle$. In the uncoupled representation

$$|l_1 l_2 m_1 m_2\rangle = |l_1 m_1\rangle |l_2 m_2\rangle = Y_{l_1}^{m_1}(\theta_1,\phi_1)\, Y_{l_2}^{m_2}(\theta_2,\phi_2)$$

There are $(2l_1+1)(2l_2+1)$ possible combinations of this kind.

Eigenstates $|lm\, l_1 l_2\rangle$ of the coupled representation are simultaneous eigenstates of the commuting operators

$$\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + 2\vec{L}_1\vec{L}_2 \qquad \hat{L}_z = \hat{L}_{1z} + \hat{L}_{2z} \qquad \hat{L}_1^2 \qquad \hat{L}_2^2$$

Any $|lm\, l_1 l_2\rangle$ can be written as a superposition of the eigenstates of the uncoupled representation:

$$|lm\, l_1 l_2\rangle = \sum_{m_1} \sum_{\substack{m_2 \\ m_1+m_2=m}} |l_1 l_2 m_1 m_2\rangle\langle l_1 l_2 m_1 m_2 | lm\, l_1 l_2\rangle$$

The constraint $m_1+m_2=m$ stems from $\hat{L}_z = \hat{L}_{1z} + \hat{L}_{2z}$ and the orthogonality of states $|l_1 l_2 m_1 m_2\rangle$

Coefficients $\langle l_1 l_2 m_1 m_2 | lm\, l_1 l_2\rangle = C_{m_1 m_2}$ are called Clebsch-Gordon coefficients. Their meaning is the following. In state $|lm\, l_1 l_2\rangle$ particles are known to have respective angular momentum quantum numbers l_1 and l_2, and total angular momentum ad z component quantum numbers l and m. The question then may be asked: what measurement L_{1z} and L_{2z} will find in state $|l m l_1 l_2\rangle$. The answer to this question is

$$|C_{m_1 m_2}|^2 = \text{probability that measurement finds one electron with } L_{1z} = m_1\hbar \text{ and the other electron with } L_{2z} = m_2\hbar$$

Illustration:

Consider the state $|l m\, l_1 l_2\rangle = |1,-1,1,1\rangle$

with $m_1+m_2=-1$ the expansion becomes

$$|1,-1,1,1\rangle = C_{0-1}|1,0\rangle|1,-1\rangle + C_{-10}|1,-1\rangle|1,0\rangle$$

Coefficents C_{0-1} and C_{-10} can be determined by normalization and application of the $\hat{L}_+$ and $\hat{L}_-$ operators. For the case at hand $\hat{L}_- = \hat{L}_{1-} + \hat{L}_{2-}$ and

$$\hat{L}_-|1,-1,1,1\rangle = 0 = (\hat{L}_{1-}+\hat{L}_{2-})\left(C_{0-1}|1,0\rangle|1,-1\rangle + C_{-10}|1,-1\rangle|1,0\rangle\right)$$

$$= \sqrt{2}\,(C_{0-1} + C_{-10})|1,-1\rangle|1,-1\rangle$$

Thus we may conclude that $C_{0-1} = -C_{-10}$. Normalization yields $C_{0-1} = \frac{1}{\sqrt{2}}$ $C_{-10} = -\frac{1}{\sqrt{2}}$

Next let us consider the problem of finding the allowed values of (l,m) given (l_1,l_2). Since $L_z = L_{1z} + L_{2z}$ it follows that

$$m_{max} = m_{1\,max} + m_{2\,max} \quad \text{or} \quad m_{max} = l_1 + l_2$$

Of the various values the total angular momentum may assume, the maximum value is equal to m_{max}. We then can deduce that

$$l_{max} = l_1 + l_2$$

In order to determine l_{min} we note the following. As mentioned previosly in the uncoupled representation there are $(2l_1+1)(2l_2+1)$ independent, common eigen-states of $\hat{L}_1^2, \hat{L}_2^2, \hat{L}_{1z}$, and $\hat{L}_{2z}$. They span $(2l_1+1)(2l_2+1)$ dimensional space. A change in the representation leaves dimensionality unchanged. This fact allows to find l_{min} such that the number of independent states equals $(2l_1+1)(2l_2+1)$:

$$\sum_{l=l_{min}}^{l_{max}} (2l+1) = (2l_1+1)(2l_2+1)$$

Since $l_{max} = l_1 + l_2$ we can find that

$$l_{min} = |l_1 - l_2|$$

Hence we have determined that for a composite system of two angular momenta l_1 and l_2 the allowed values of l are

$$l = |l_1-l_2|, |l_1-l_2|+1, \ldots, l_1+l_2$$