# Electron in magnetic field

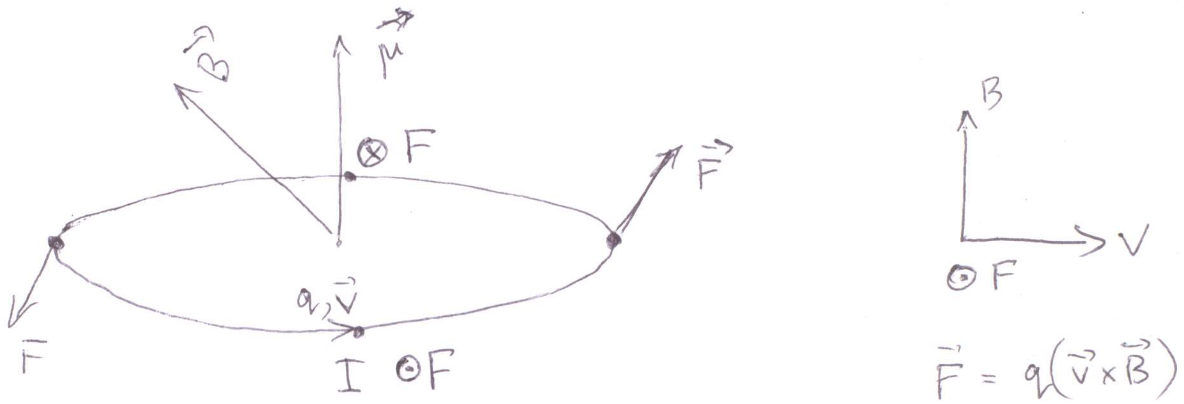
In classical mechanics a spinning charged particle forms a magnetic dipole. This magnetic dipole is proportional to to its spin angular momentum:

$$\vec{\mu} = \gamma \vec{S}$$

where $\gamma$ is a constant that depends on the magnitude and sign of the charge.

A similar relationship takes place in quantum mechanics. Constant $\gamma$ is called the gyromagnetic ratio and is equal $\gamma = -\frac{e}{m}$ (SI or Gauss)

When a magnetic dipole is placed in a magnetic field $\vec{B}$, it experiences a torque, $\vec{\mu} \times \vec{B}$, which tends to line it up parallel to the $\vec{B}$ field.




The energy associated with this torque is $H = -\vec{\mu} \cdot \vec{B}$. Hence, the Hamiltonian of a particle with spin in a magnetic field becomes

$$\hat{H} = -\gamma \vec{B} \cdot \vec{S}$$

## Larmor precession

Consider a particle of spin 1/2 at rest in a uniform magnetic field, which points in the z-direction

$$\vec{B} = B\,\vec{e}$$

The interaction of the particle with this field is described by the Hamiltonian

$$\hat{H} = -\gamma \vec{B}\cdot\hat{\vec{S}} = -\gamma B \hat{S}_z = -\frac{\gamma B \hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

The eigenstates of $\hat{H}$ are the same as those of $\hat{S}_z$:

$$E_+ = -\frac{\gamma B \hbar}{2} \qquad \chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$$

$$E_- = \frac{\gamma B \hbar}{2} \qquad \chi_- = \begin{pmatrix}0\\1\end{pmatrix}$$

Energy is lowest when the dipole is parallel to the $\vec{B}$ field (i.e. when its projection on the $\vec{B}$ axis is positive)

Now let us see how the spin state of the particle evolves with time. The general solution of the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \chi}{\partial t} = \hat{H}\chi \qquad \chi \equiv \begin{pmatrix}f(t)\\ g(t)\end{pmatrix}$$

can be expressed in terms of the stationary states:

$$\chi(t) = a\chi_+ e^{-\frac{iE_+t}{\hbar}} + b\chi_- e^{-\frac{iE_-t}{\hbar}} = \begin{pmatrix}a e^{\frac{i\gamma B t}{2}}\\ b e^{-\frac{i\gamma B t}{2}}\end{pmatrix}$$

at $t=0$

$$\chi(0) = \begin{pmatrix}a\\b\end{pmatrix} \qquad \text{and} \qquad |a|^2 + |b|^2 = 1$$

We can write $a$ and $b$ as

$$a = \cos\frac{\alpha}{2} \qquad b = \sin\frac{\alpha}{2} \qquad (\text{so that } |a|^2+|b|^2=1)$$

Then

$$\chi(t) = \begin{pmatrix} \cos\frac{\alpha}{2}\, e^{\frac{i\gamma B t}{2}} \\ \sin\frac{\alpha}{2}\, e^{-\frac{i\gamma B t}{2}} \end{pmatrix}$$

Now let us compute $\langle S_x \rangle$, $\langle S_y \rangle$, and $\langle S_z \rangle$

$$\langle S_x \rangle = \chi^+(t)\, S_x\, \chi(t) = \left(\cos\frac{\alpha}{2}\, e^{-\frac{i\gamma B t}{2}},\ \sin\frac{\alpha}{2}\, e^{\frac{i\gamma B t}{2}}\right) \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2}\, e^{\frac{i\gamma B t}{2}} \\ \sin\frac{\alpha}{2}\, e^{-\frac{i\gamma B t}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{2} \sin\alpha \cos(\gamma B t)$$

Similarly

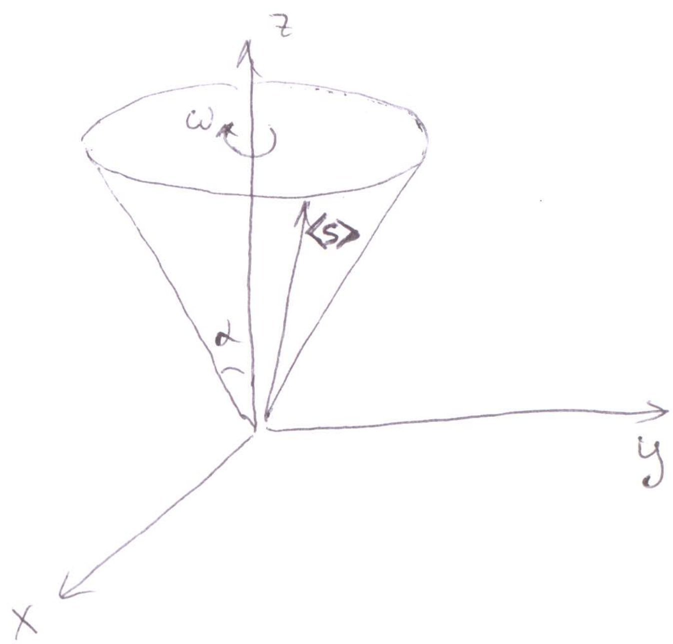
$$\langle S_y \rangle = \chi^+(t)\, S_y\, \chi(t) = -\frac{\hbar}{2} \sin\alpha \sin(\gamma B t)$$

Lastly

$$\langle S_z \rangle = \chi^+(t)\, S_z\, \chi(t) = \frac{\hbar}{2}\left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right) = \frac{\hbar}{2}\cos\alpha$$

We can see that $\langle \vec{S} \rangle$ precesses about the z-axis ($\vec{B}$ direction) with the Larmor frequency:

$$\omega = \gamma B$$



# Stern - Gerlach experiment

The energy associated with a magnetic dipole in magnetic field is $-\vec{\mu}\cdot\vec{B}$. If $\vec{\mu}$ is not aligned with the direction of $\vec{B}$ then there is a torque that tries to align $\vec{\mu}$ with $\vec{B}$. If $\vec{B}$ is uniform then the net force vanishes because $\vec{F} = -\frac{\partial V}{\partial \vec{r}} = 0$ $(V = -\vec{\mu}\vec{B})$. In inhomogeneous magnetic field, however, this force is no longer zero. In fact, it can be used to separate out particles with a particular spin orientation.

Let' us assume that we have a situation when heavy neutral atoms traveling in y-direction enter a region of weak inhomogeneity so that the field is not $B\vec{e}_z$ but $\vec{B}(x,y,z) = -\alpha x\,\vec{e}_x + (B+\alpha z)\,\hat{e}_z$, where $\alpha$ is small. The distortion just along z axis is impossible because $\nabla_{\vec{r}}\cdot\vec{B} = 0$. For this reason we have to have the field distorted along both x and z axis.

The force is then

$$\vec{F} = -\nabla\left(-\vec{\mu}\cdot\vec{B}\right) = -\nabla\left(-\gamma\vec{S}\cdot\vec{B}\right) =$$

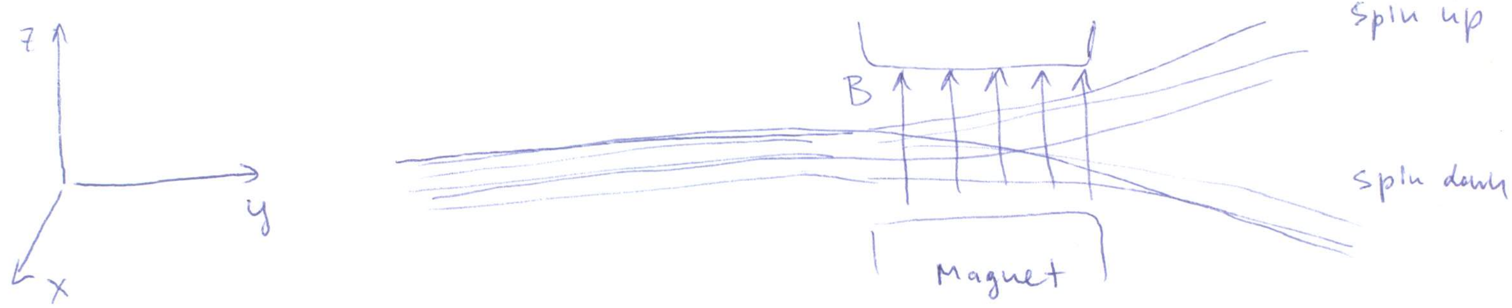
$$= \gamma\alpha\left(-S_x\vec{e}_x + S_z e_z\right)$$

Now from the previous lecture we know that the x-component of spin oscillates rapidly (with frequency $\omega = \gamma B$). Therefore the x-component of the force averages to zero. The z-component, $F_z$, does not vanish because $\langle S_z\rangle$ is constant (i.e. does not oscillate):

$$F_z = \gamma\alpha S_z$$

Therefore, particles are pulled up or down (in the z-direction) depending on the z-component of spin. For classical particles we would observe a smear (because $S_z$ can take any value from $-S$ to $+S$). For quantum particles, however, $S_z$ is quantized. In fact the beam of particles with angular momentum $S$ splits into $2s+1$ separate streams.

In case of atoms this splitting effect comes mainly from unpaired electron(s). This is because it is proportional to $\gamma = \frac{q}{m}$ and the mass of protons and neutrons exceeds that of electrons by orders of magnitude. For hydrogen and alkali atoms the beam splits into just two streams




We can also show that a beam of spin particles is split indeed in a little more rigorous way that does not invoke classical concepts (e.g. force). Consider the process in a reference frame that moves along the y-axis with the beam. In this frame the Hamiltonian starts out zero, turns on for time $T$ (time necessary to pass the magnet), and then turs off again:

$$H(t) = \begin{cases} 0 & t < 0 \\ -\gamma(B + \alpha z) S_z & 0 \le t \le T \\ 0 & t > T \end{cases}$$

above we ignored the x-component of B because of the reasons outlined previously.

Suppose an atom with a single electron is in state at $t=0$

$$\chi(t=0) = a\chi_+ + b\chi_-$$

while the Hamiltonian acts, $\chi(t)$, evolves in the usual way (from $t=0$ to $t=T$ $\hat{H}$ is time-independent)

$$\chi(t) = a\chi_+ e^{-\frac{iE_+t}{\hbar}} + b\chi_- e^{-\frac{iE_-t}{\hbar}} \qquad 0<t<T$$

where $E_\pm = \mp\gamma(B+\alpha z)\frac{\hbar}{2}$ $\left(\text{recall that } H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\hbar}{2}\gamma(B+\alpha z)\right)$

at $t=T$ it becomes

$$\chi(t=T) = a\chi_+ e^{\frac{i\gamma TB}{2}} e^{\frac{i\alpha\gamma T}{2}z} + b\chi_- e^{-\frac{i\gamma TB}{2}} e^{-\frac{i\alpha\gamma T}{2}z}$$

We can now see that each term carry momentum in the z-direction: Indeed the plane waves are eigenfunctions of the $\hat{p}_z$ operator corresponding to the eigenvalue

$$p_z = \pm\frac{\alpha\gamma T\hbar}{2}$$

Hence, after exiting the magnet, the spin-up component acquires a momentum in the positive z-direction while the spin-down component acquires a momentum in the opposite direction.

The Stern-Gerlach experiment played an important role for the development of quantum theory. It demonstrated that particles possess an intrinsic angular momentum and this momentum takes only discrete projections on any axis.