① In the limit $n \to \infty$ the quantum probability distribution function $P_n(x)$ approaches the classical probability distribution function $p(x)$. The latter is

$p(x) \propto \frac{1}{|v(x)|}$ and $v(x)$ is the velocity of the particle

from the conservation of energy we have

$$\frac{mv^2}{2} + \gamma|x| = E \qquad v(x) = \sqrt{\frac{2}{m}(E-\gamma|x|)} \qquad |x| \le \frac{E}{\gamma}$$

Hence

$$p(x) = \begin{cases} \dfrac{c}{\sqrt{\frac{E}{\gamma} - |x|}} & , |x| \le \frac{E}{\gamma} \\ 0 & , |x| > \frac{E}{\gamma} \end{cases} \qquad c = \text{const}$$

Let us denote $a \equiv \frac{E}{\gamma}$. $p(x)$ must be normalized:

$$1 = c \int_{-a}^{a} \frac{dx}{\sqrt{a-|x|}} = 2c \int_0^a \frac{dx}{\sqrt{a-x}} = 4c\sqrt{a} \Rightarrow c = \frac{1}{4\sqrt{a}}$$

The expectation values of x and x^2 are:

$$\langle x \rangle = \frac{1}{4\sqrt{a}} \int_{-a}^{a} \frac{x}{\sqrt{a-|x|}} dx = 0 \qquad \text{(odd integrand)}$$

$$\langle x^2 \rangle = \frac{1}{4\sqrt{a}} \int_{-a}^{a} \frac{x^2}{\sqrt{a-|x|}} dx = \frac{1}{2\sqrt{a}} \int_0^a \frac{x^2}{\sqrt{a-x}} dx \xlongequal{\text{some manipulations}}$$

$$= \frac{1}{2\sqrt{a}} \left(-\frac{2}{15}\sqrt{a-x}\,(8a^2 + 4ax + 3x^2) \right) \Big|_0^a = \frac{8}{15} a^2 = \frac{8}{15} \frac{E_n^2}{\gamma^2}$$

② The momentum space wave function is

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x) e^{-ikx} dx = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\beta - x^2) e^{-\frac{x^2}{2\beta} - ikx} dx$$

Here we can use the values of the following integrals:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2 - ikx} dx = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}} \qquad \left(\text{in our case } \alpha \equiv \frac{1}{2\beta}\right)$$

$$\int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2 - ikx} dx = \left(-\frac{\partial}{\partial\alpha}\right) \int_{-\infty}^{+\infty} e^{-\alpha x^2 - ikx} dx = \left(-\frac{\partial}{\partial\alpha}\right) \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}} = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{k^2}{4\alpha}} \left(\frac{1}{2\alpha} - \frac{k^2}{4\alpha^2}\right)$$

Then

$$\tilde{\psi}(k) = \frac{A}{\sqrt{2\pi}} \left(\beta - \left(\frac{2\beta}{2} - \frac{k^2 4\beta^2}{4}\right)\right) \sqrt{2\pi\beta}\, e^{-\frac{\beta k^2}{2}} = A \beta^{5/2} k^2 e^{-\frac{\beta k^2}{2}}$$

The maximum of $p(k) = |\tilde{\psi}(k)|^2 = A^2 \beta^5 k^4 e^{-\beta k^2}$ can be easily found:

$$\frac{\partial |\tilde{\psi}|^2}{\partial k} = 0 \quad \Longrightarrow \quad (4k^3 - 2k^5 \beta) = 0 \quad \Longrightarrow \quad K_{most.prob} = \pm\sqrt{\frac{2}{\beta}}$$

Therefore the most probable value of the momentum is

$$P_{most.prob} = \hbar k_{most.prob} = \pm \hbar \sqrt{\frac{2}{\beta}}$$

③ a) $\hat{T}_a = e^{-\frac{i\hat{p}a}{\hbar}} = e^{-a\frac{d}{dx}}$

$$\hat{T}_a^{+} = \left(e^{-\frac{i\hat{p}a}{\hbar}}\right)^{+} = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{i\hat{p}a}{\hbar}\right]^n\right)^{+} = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{i\hat{p}a}{\hbar}\right]^n\right) = e^{\frac{i\hat{p}a}{\hbar}} = \hat{T}_{-a}$$

Note that for an arbitrary function $\hat{T}_a f(x) = f(x-a)$.
Therefore,

$$\left(\hat{T}_a^{+} V(x)\hat{T}_a\right)f(x) = \hat{T}_{-a}V(x)T_a f(x) = T_{-a}V(x)f(x-a) = V(x+a)f(x)$$

So $\quad \hat{T}_a^{+}V(x)\hat{T}_a = V(x+a)$

b) $[\hat{T}_a, V(x)]f(x) = \hat{T}_a V(x)f(x) - V(x)\hat{T}_a f(x) = V(x-a)f(x-a) - V(x)f(x-a) =$

$$= V(x-a)\hat{T}_a f(x) - V(x)\hat{T}_a f(x) = \left(V(x-a) - V(x)\right)T_a f(x)$$

So

$$[\hat{T}_a, V(x)] = \left(V(x-a)-V(x)\right)\hat{T}_a$$

c) The rate of change of $\langle p\rangle$ is given by the generalized Ehrenfest theorem:

$$\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}[\hat{H},\hat{p}] + \underbrace{\left\langle\frac{\partial\hat{p}}{\partial t}\right\rangle}_{0,\ \text{because } \hat{p}\neq\hat{p}(t)}$$

Invariance under arbitrary translations implies that
$[\hat{H},\hat{T}_a] = 0 \quad$ or $\quad \hat{T}_a\hat{H} = \hat{H}\hat{T}_a$

Equivalently we can just say that $T_a V(x) = V(x-a) = V(x)$ for any value of a. This means V must be a constant

Then

$$[\hat{H},\hat{p}] = \left[\frac{\hat{p}^2}{2m} + V(x), \hat{p}\right] = [\underbrace{V(x)}_{\text{const}}, \hat{p}] = 0$$

Therefore,

$$\frac{d\langle p\rangle}{dt} = 0 \quad \Longrightarrow \quad \langle p\rangle = \text{const}$$

(4) a) The ground state wave function ($Z=2$) is

$$\psi(r,\theta,\phi) = R_{10}(r)\, Y_0^0(\theta,\phi) = \frac{2}{a^{3/2}} e^{-\frac{r}{a}} \frac{1}{\sqrt{4\pi}} \qquad \text{where } a \equiv \frac{a_0}{Z}$$

The radial probability distribution function is

$$\rho(r) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta\, |\psi|^2 = \frac{4}{a^3} e^{-\frac{2r}{a}} r^2$$

Most probable value or r corresponds to the maximum of $\rho(r)$:

$$\frac{\partial \rho}{\partial r} = 0 \implies \left(2r - \frac{2}{a} r^2\right) = 0 \implies r = a$$

b)

$$P(0<r<b) \simeq |\psi(0)|^2 \cdot \underbrace{\frac{4}{3}\pi b^3}_{\text{volume of the nucleus}} = \frac{4}{a^3} \frac{1}{4\pi} \cdot \frac{4}{3}\pi b^3 = \frac{4}{3}\frac{b^3}{a^3} \qquad b \ll a$$

c)

$$P = |\langle \psi_{100}^{new} | \psi_{100}^{old} \rangle|^2 = \left| \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} r^2 dr \cdot \psi_{100}^{new}(\vec{r})\, \psi_{100}^{old}(\vec{r}) \right|^2 =$$

$$= \left| \int_0^{\infty} r^2 \frac{2}{a_{new}^{3/2}} e^{-\frac{r}{a_{new}}} \frac{2}{a_{old}^{3/2}} e^{-\frac{r}{a_{old}}} dr \right|^2 = \left| \frac{4}{a_{new}^{3/2} a_{old}^{3/2}} \int_0^{\infty} e^{-\beta r} r^2 dr \right|^2 \qquad \beta = \frac{1}{a_{new}} + \frac{1}{a_{old}} = \frac{a_{new} + a_{old}}{a_{new} a_{old}}$$

$$= \left| \frac{4}{a_{new}^{3/2} a_{old}^{3/2}} \frac{2}{\beta^3} \right|^2 = \left| 8 \frac{a_{new}^{3/2} a_{old}^{3/2}}{(a_{new} + a_{old})^3} \right|^2 = 64 \left[\frac{\sqrt{a_{new} a_{old}}}{a_{new} + a_{old}} \right]^6 =$$

$$= 64 \left[\frac{\sqrt{a_0 \cdot 2a_0}}{a_0 + 2a_0} \right]^6 = 64 \left[\frac{\sqrt{2}}{3} \right]^6 = \frac{512}{729} \approx 0.702$$