Probability current

Let us consider the time derivative of particle density $\rho(x,t) = |\psi(x,t)|^2$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\psi^*\psi = \frac{\partial \psi^*}{\partial t}\psi + \psi^*\frac{\partial \psi}{\partial t}$$

Quantities $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi^*}{\partial t}$ can be taken from the time-dependent Schrödinger equation and its complex conjugate:

$$\frac{\partial \psi}{\partial t} = \frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right)$$

$$\frac{\partial \psi^*}{\partial t} = \frac{1}{-i\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V\psi^*\right)$$

When we substitute these into the expression for $\frac{\partial \rho}{\partial t}$ the terms containing V will cancel out (we assume $V^* = V$) and we get

$$\frac{\partial \rho}{\partial t} = \frac{-i\hbar}{2m}\left(\psi\frac{\partial^2 \psi^*}{\partial x^2} - \psi^*\frac{\partial^2 \psi}{\partial x^2}\right)$$

We can rewrite it as

$$\frac{\partial \rho}{\partial t} = \frac{-i\hbar}{2m}\frac{\partial}{\partial x}\left(\psi\frac{\partial \psi^*}{\partial x} - \psi^*\frac{\partial \psi}{\partial x}\right)$$

or

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} j$$

where

$$j = \frac{i\hbar}{2m}\left(\psi\frac{\partial \psi^*}{\partial x} - \psi^*\frac{\partial \psi}{\partial x}\right) = \frac{\hbar}{m}\operatorname{Im}\left[\psi^*\frac{\partial \psi}{\partial x}\right] = \frac{1}{m}\operatorname{Re}\left[\psi^*\hat{p}\psi\right]$$

If we did the same exercise in 3D by replacing

$$\frac{\partial^2 \psi}{\partial x^2} \longrightarrow \nabla^2 \psi .$$

we would have obtained

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{j} \qquad \text{where} \quad \vec{j} = \frac{i\hbar}{2m}\left(\psi \nabla \psi^* - \psi^* \nabla \psi\right)$$

$$= \frac{\hbar}{m} \operatorname{Im}\left[\psi^* \nabla \psi\right]$$

$$= \frac{1}{m} \operatorname{Re}\left[\psi^* \hat{\vec{p}}\, \psi\right]$$

The above equations look just like the familiar continuity equation where j (or $\vec{j}$) is a flux. This flux can be associated with the "probability current".

For charged particles (e.g. electron) the probability density is proportional to the charge density.

$$\rho_{\text{charge}} = \underbrace{q}_{\text{charge}} \rho_{\text{prob.}}$$

Similarly the probability current is pretty much the same (up to a constant factor q) as the charge current.