Exchange interaction

The requirement of (anti)symmetry of the wave function for identical particles leads to some important implications. One of them is the Pauli exclusion principle, as we saw it previously. Moreover, some of the implications may occur for macroscopic systems. For example, the well-known Bose-Einstein condensation of bosons at low temperature results from the symmetry of their wave function.

When we talked about systems of bosons or fermions we assumed that the particles do not interact with each other. Strictly speaking this cannot be an absolutely accurate assumption. If particles are in proximity of each other so that their wavefunctions overlap the (anti)symmetry requirement leads to some "effective" "interaction" coming from the indistinguishability of particles. This "interaction", however, is not like what we are used to think about. It cannot be described by a simple function $V(\vec{r}_1, \vec{r}_2)$. The nature of this interaction is more subtle. We will try to understand it using a simple one-dimensional example.

Suppose we have two particles in states $\psi_a(x)$ and $\psi_b(x)$ and let us assume that the states are orthogonal. For distinguishable noninteracting particles the total wave function is

$$\psi(x_1, x_2) = \psi_a(x_1)\,\psi_b(x_2)$$

For indistinguishable particles, on the other hand,

$$\psi_\pm(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)\right)$$ + for bosons, − for fermions

Let us calculate the expectation value $\langle (x_1 - x_2)^2 \rangle$ for distinguishable particles

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

$$\langle x_1^2 \rangle = \int x_1^2 |\psi_a(x_1)|^2 dx_1 \int |\psi_b(x_2)|^2 dx_2 = \langle x^2 \rangle_a$$

$$\langle x_2^2 \rangle = \int |\psi_a(x_1)|^2 dx_1 \int x_2^2 |\psi_b(x_2)|^2 dx_2 = \langle x^2 \rangle_b$$

and

$$\langle x_1 x_2 \rangle = \int x_1 |\psi_a(x_1)|^2 dx_1 \int x_2 |\psi_b(x_2)|^2 dx_2 = \langle x \rangle_a \langle x \rangle_b$$

Thus

$$\langle (x_1 - x_2)^2 \rangle = \langle x \rangle_a^2 + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b$$

Note that the answer is actually independent on which particle is in state ψ_a and which is in state ψ_b.

Now let us compute the same expectation value for the case when particles are identical

$$\langle x_1^2 \rangle = \frac{1}{2}\left(\iint x_1^2 \left|\left[\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)\right]\right|^2 dx_1 dx_2 \right) =$$

$$= \frac{1}{2}\Big(\int x_1^2 |\psi_a(x_1)|^2 dx_1 \int |\psi_b(x_2)|^2 dx_2 + \int x_1^2 |\psi_b(x_1)|^2 dx_1 \int |\psi_a(x_2)|^2 dx_2 +$$

$$\pm \int x_1^2 \psi_a^*(x_1)\psi_b(x_1) dx_1 \int \psi_b^*(x_2)\psi_a(x_2) dx_2$$

$$\pm \int x_1^2 \psi_b^*(x_1)\psi_a(x_1) dx_1 \int \psi_a^*(x_2)\psi_b(x_2) dx_2 \Big) = \frac{1}{2}\left[\langle x^2 \rangle_a + \langle x^2 \rangle_b \pm 0 \pm 0\right]$$

$$= \frac{1}{2}\left[\langle x^2 \rangle_a + \langle x^2 \rangle_b\right]$$

Similarly we can obtain $\langle x_2^2 \rangle$:

$$\langle x_2^2 \rangle = \frac{1}{2}\left(\langle x^2\rangle_b + \langle x^2\rangle_a\right)$$

$$\langle x_1 x_2 \rangle = \frac{1}{2}\Big(\int x_1 |\psi_a(x_1)|^2 dx_1 \int x_2 |\psi_b(x_2)|^2 dx_2 + \int x_1 |\psi_b(x_1)|^2 dx_1 \int x_2 |\psi_a(x_2)|^2 dx_2$$

$$\pm \int x_1 \psi_a^*(x_1) \psi_b(x_1)\, dx_1 \int x_2 \psi_b^*(x_2)\psi_a(x_2)\, dx_2$$

$$\pm \int x_1 \psi_b^*(x_1)\psi_a(x_1)\, dx_1 \int x_2 \psi_a^*(x_2)\psi_b(x_2)\, dx_2 \Big)$$

$$= \frac{1}{2}\left(\langle x\rangle_a \langle x\rangle_b + \langle x\rangle_b\langle x\rangle_a \pm \langle x\rangle_{ab}\langle x\rangle_{ba} \pm \langle x\rangle_{ba}\langle x\rangle_{ab}\right) =$$

$$= \langle x\rangle_a \langle x\rangle_b \pm |\langle x\rangle_{ab}|^2$$

In the end we get

$$\langle (x_1 - x_2)^2\rangle_\pm = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b \mp 2|\langle x\rangle_{ab}|^2$$

The difference between distinguishable and nondistinguishable particles case is then $(\Delta x \equiv x_1 - x_2)$

$$\langle (\Delta x)^2\rangle_\pm = \langle (\Delta x)^2\rangle_d \mp 2|\langle x\rangle_{ab}|^2$$

Identical bosons tend to get somewhat closer together while fermions tend to get father apart.

If ψ_a and ψ_b are well separated in space so that their overlap $\left(\int \psi_a^*(x)\psi_b(x)\,dx\right)$ vanishes then the "exchange" effect disappears.

The tendency of identical particles to get closer or farther apart gives rise to so called exchange forces.