The Heisenberg uncertainty principle reads

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

Therefore we will need to compute $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, and $\langle p^2 \rangle$. In the process of evaluating those we will encounter two integrals:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \frac{\pi^{1/2}}{\alpha^{1/2}} \quad \text{and} \quad \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} dx = -\frac{\partial}{\partial \alpha} \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \frac{1}{2} \frac{\pi^{1/2}}{\alpha^{3/2}}$$

Constant C can be determined from the normalization:

$$1 = \int_{-\infty}^{+\infty} |C|^2 e^{-\alpha x^2} dx = |C|^2 \frac{\pi^{1/2}}{\alpha^{1/2}} \implies C = \frac{\alpha^{1/4}}{\pi^{1/4}}$$

Then we have

$$\langle x \rangle = \int_{-\infty}^{+\infty} x |\psi_0|^2 dx = 0 \quad \text{(odd integrand)}$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 |\psi_0|^2 dx = \frac{\alpha^{1/2}}{\pi^{1/2}} \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} dx = \frac{\alpha^{1/2}}{\pi^{1/2}} \frac{1}{2} \cdot \frac{\pi^{1/2}}{\alpha^{3/2}} = \frac{1}{2\alpha}$$

$$\langle p \rangle = \int_{-\infty}^{+\infty} \psi_0^* \left(-i\hbar \frac{d}{dx} \psi_0\right) dx = 0 \quad \text{(odd integrand)}$$

$$\langle p^2 \rangle = \int_{-\infty}^{+\infty} \psi_0^2 \left(-\hbar^2 \frac{d^2}{dx^2} \psi_0\right) dx = -\hbar^2 \int_{-\infty}^{+\infty} \frac{\alpha^{1/4}}{\pi^{1/4}} e^{-\frac{\alpha x^2}{2}} \left(\frac{d}{dx} (-\alpha x) e^{-\frac{\alpha x^2}{2}}\right) dx =$$

$$= -\hbar^2 \frac{\alpha^{1/4}}{\pi^{1/4}} \int_{-\infty}^{+\infty} e^{-\frac{\alpha x^2}{2}} (\alpha^2 x^2 - \alpha) e^{-\frac{\alpha x^2}{2}} dx = -\hbar^2 \frac{\alpha^{1/4}}{\pi^{1/4}} \left(\alpha^2 \frac{1}{2} \frac{\pi^{1/2}}{\alpha^{3/2}} - \alpha \frac{\pi^{1/2}}{\alpha^{1/2}}\right) = \frac{\hbar^2 \alpha}{2}$$

All this yields

$$\Delta x \cdot \Delta p = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{1}{2\alpha}} \cdot \sqrt{\frac{\hbar^2 \alpha}{2}} = \frac{\hbar}{2}$$

Hence the uncertainty principle, $\Delta p \Delta x \geq \frac{\hbar}{2}$, holds true.