1. Our trial wave function has the following form:

$$\psi(r) = A e^{-\alpha r}$$

The normalization integral is:

$$\langle\psi|\psi\rangle = \underbrace{4\pi}_{\text{integral over angles } \theta \text{ and } \phi} \cdot |A|^2 \underbrace{\int_0^\infty e^{-2\alpha r} r^2 dr}_{\frac{2!}{(2\alpha)^3}} = \frac{\pi |A|^2}{\alpha 3}$$

Kinetic energy integral ($\ell = 0$)

$$\langle\psi|T|\psi\rangle = \langle\psi| -\frac{\hbar^2}{2m}\nabla^2|\psi\rangle = \langle\psi| -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)|\psi\rangle =$$

$$= -\frac{\hbar^2}{2m}|A|^2 \cdot 4\pi \cdot \int_0^\infty e^{-\alpha r}\left(\alpha^2 - \frac{2\alpha}{r}\right) e^{-\alpha r} r^2 dr =$$

$$= -\frac{\hbar^2}{2m}|A|^2 \cdot 4\pi \left(\frac{\alpha^2 \cdot 2}{(2\alpha)^3} - \frac{2\alpha}{(2\alpha)^2}\right) = \frac{\hbar^2 |A|^2 \pi}{2m\alpha}$$

Potential energy integral

$$\langle\psi|V|\psi\rangle = \langle\psi| -g e^{-2br}|\psi\rangle = -|A|^2 g \cdot 4\pi \int_0^\infty e^{-2\alpha} e^{-2b} r^2 dr =$$

$$= -|A|^2 g \cdot 4\pi \frac{2}{(2(\alpha+b))^3} = -\frac{|A|^2 g \pi}{(\alpha+b)^3}$$

Variational energy:

$$E = \frac{\langle\psi|T+V|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2\alpha^2}{2m} - \frac{g\alpha^3}{(\alpha+b)^3}$$

Now let us minimize E with respect to parameter α:

$$\frac{\partial E}{\partial \alpha} = \frac{\hbar^2}{m}\alpha - \frac{3g\alpha^2}{(\alpha+b)^3} + \frac{3g\alpha^3}{(\alpha+b)^4} = \frac{\hbar^2}{m}\alpha + \frac{-3g\alpha^2(\alpha+b) + 3g\alpha^3}{(\alpha+b)^4} =$$

$$= \frac{\hbar^2}{m}\alpha - \frac{3gb\alpha^2}{(\alpha+b)^4} = 0$$

or

$$\alpha\left[(\alpha+b)^4 - \frac{3gbm}{\hbar^2}\alpha\right] = 0$$

$\alpha = 0$ is a trivial solution that corresponds to an unbound state (think of $e^{-\alpha r}$).

The solution of the equation

$$(\alpha+b)^4 - \frac{3gbm}{\hbar^2}\alpha = 0$$

has real roots only when $\frac{3gbm}{\hbar^2} > \frac{256}{27} b^3$

or $g > \frac{256}{81}\frac{b^2\hbar^2}{m}$

Thus , if $g > \frac{256}{81}\frac{b^2\hbar^2}{m}$ then the potential definitely supports bound states

2. a) Π obviously commutes with H^0, H', and H because all of them are invariant under $x \rightleftarrows y$ permutation. Hence H and Π have a mutual system of eigenfunctions that satisfy the property

$$\psi(x,y) = \pm\psi(y,x)$$

b) The ground state of the system is

$$|00\rangle \equiv |0\rangle|0\rangle \equiv |\phi_0(x)\rangle|\phi_0(y)\rangle$$

where ϕ_0 is the ground state of 1D harmonic oscillator. This state, $|00\rangle$, is nondegenerate, because

$$E^{(0)}_{00} = \hbar\omega(\underbrace{0}_{n_x} + \underbrace{0}_{n_y} + 1) = \hbar\omega$$

can only be achieved with a single combination of quantum numbers n_x and n_y (both are zero). Therefore, the first order correction is:

$$E^{(1)}_{00} = \langle 00|H'|00\rangle = 2\lambda\langle 0|x^2|0\rangle\langle 0|y^2|0\rangle = 2\lambda\frac{\hbar}{2m\omega}\cdot\frac{\hbar}{2m\omega} =$$

$$= \frac{\lambda}{2}\frac{\hbar^2}{m^2\omega^2}$$

c) The first order correction to the wave function is

$$\psi^{(1)}_{00} = \sum_{\substack{m_x, m_y \\ m_x m_y \neq 00}} \frac{\langle m_x m_y|H'|00\rangle}{E^{(0)}_{00} - E^{(0)}_{m_x m_y}} |m_x m_y\rangle$$

In our case most matrix elements will be zero, except

$$\langle 20|H'|00\rangle = 2\lambda\langle 2|x^2|0\rangle\langle 0|y^2|0\rangle = 2\lambda\cdot\frac{\hbar}{2m\omega}\sqrt{2}\cdot\frac{\hbar}{2m\omega} = \frac{\lambda}{\sqrt{2}}\frac{\hbar^2}{m^2\omega^2}$$

$$\langle 02|H'|00\rangle = 2\lambda\langle 0|x^2|0\rangle\langle 2|y^2|0\rangle = \frac{\lambda}{\sqrt{2}}\frac{\hbar^2}{m^2\omega^2}$$

and

$$\langle 22|H'|00\rangle = 2\lambda \langle 2|x^2|0\rangle\langle 2|y^2|0\rangle = \lambda \frac{\hbar^2}{m^2\omega^2}$$

With that we have

$$\psi_{00}^{(1)} = \frac{\frac{\lambda}{\sqrt{2}}\frac{\hbar^2}{m^2\omega^2}}{\hbar\omega - 3\hbar\omega}|20\rangle + \frac{\frac{\lambda}{\sqrt{2}}\frac{\hbar^2}{m^2\omega^2}}{\hbar\omega - 3\hbar\omega}|02\rangle + \frac{\lambda\frac{\hbar^2}{m^2\omega^2}}{\hbar\omega - 5\hbar\omega}|22\rangle$$

$$= -\frac{\lambda\hbar}{4m^2\omega^3}\left(\sqrt{2}|20\rangle + \sqrt{2}|02\rangle + |22\rangle\right)$$

{ this expression is symmetric under $x \rightleftarrows y$ permutation

d) The first excited energy level is doubly degenerate (01 & 10) so we need to use the degenerate perturbation theory. Operator Π commutes with H. Hence, the eigenstates of Π should form a proper basis, in which H is diagonal. The eigenstates of Π made of $|10\rangle$ and $|01\rangle$ are

$$\varphi_1 = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \qquad \varphi_2 = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$$

$$\Pi\varphi_{1,2} = \pm\varphi_{1,2}$$

In this basis we only need to compute $\langle\varphi_1|H'|\varphi_1\rangle$ and $\langle\varphi_2|H'|\varphi_2\rangle$ to obtain the first order corrections.

$$\langle\varphi_1|H'|\varphi_1\rangle = \frac{1}{2}\Big[\underbrace{\langle 10|H'|10\rangle}_{\frac{3}{2}\frac{\hbar^2}{m^2\omega^2}} + \underbrace{\langle 10|H'|01\rangle}_{0} + \underbrace{\langle 01|H'|10\rangle}_{0} + \underbrace{\langle 01|H'|01\rangle}_{\frac{3}{2}\frac{\hbar^2}{m^2\omega^2}}\Big]$$

$$= \frac{3\lambda}{2}\frac{\hbar^2}{m^2\omega^2}$$

$$\langle\varphi_2|H'|\varphi_2\rangle = \frac{1}{2}\Big[\langle 10|H'|10\rangle - \langle 10|H'|01\rangle - \langle 01|H'|10\rangle + \langle 01|H'|01\rangle\Big] =$$

$$= \frac{3\lambda}{2}\frac{\hbar^2}{m^2\omega^2}$$

The first excited energy level remains degenerate to first order with an energy correction $E^{(1)} = \frac{3\lambda}{2}\frac{\hbar^2}{m^2\omega^2}$

3.

a) The wave functions are

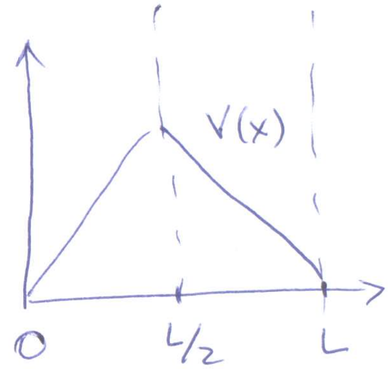
$$\psi_\pm(x) = \frac{C_\pm}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar}\int p(x)\,dx} \qquad \text{where } p(x)=\sqrt{2m[E-V(x)]}$$

and each energy is two-fold degenerate

b) Since the particle is on a ring, the wave function must be periodic, $\psi(0)=\psi(L)$

$$\frac{1}{\hbar}\int_0^L \sqrt{2m[E-V(x)]}\,dx = 2\pi n \qquad n=\pm1,\pm2,\ldots \quad (n=0 \text{ is not possible})$$

c) Here we need to use the above quantization condition



$$\int_0^L \sqrt{E_n - V(x)}\,dx = 2\int_0^{L/2}\sqrt{E_n-V(x)}\,dx = 2\int_0^{L/2}\sqrt{E_n-\alpha x}\,dx =$$

$$= 2\alpha^{1/2}\int_0^{L/2}\left(\frac{E_n}{\alpha}-x\right)^{1/2}dx = -2\alpha^{1/2}\frac{2}{3}\left(\frac{E_n}{\alpha}-x\right)\Big|_0^{L/2} = -\frac{4\alpha^{1/2}}{3}\left[\left(\frac{E_n}{\alpha}-\frac{L}{2}\right)^{3/2}-\left(\frac{E_n}{\alpha}\right)^{3/2}\right]$$

$$= \frac{4}{3\alpha}\left[E_n^{3/2} - \left(E_n - \frac{L\alpha}{2}\right)^{3/2}\right]$$

Then the energies are found as the roots of the following equation

$$E_n^{3/2} - \left(E_n - \frac{L\alpha}{2}\right)^{3/2} = \frac{2\pi\hbar n}{\sqrt{2m}}$$

The analytic solution is possible but results in a complicated expression.

4. In the first order time-dependent perturbation theory we have (for $n \neq 1$):

$$c_n^{(1)}(t=\infty) = \frac{1}{i\hbar}\int_0^\infty \langle n|H'(t')|1\rangle e^{i\omega_{n1}t'}dt' = \frac{\lambda}{i\hbar}\int_0^\infty \langle n|x|1\rangle e^{-\frac{t'}{\tau}} e^{i\omega_{n1}t'}dt'$$

where $\omega_{n0} = \dfrac{E_n^{(0)} - E_1^{(0)}}{\hbar} = \dfrac{\hbar\pi^2}{2ma^2}(n^2-1)$

$$\langle n|x|1\rangle \overset{n\neq 1}{=} \begin{cases} 0, & n \text{ is odd} \\ -\dfrac{8na}{\pi^2(n^2-1)^2}, & n \text{ is even} \end{cases}$$

The integral over t' is

$$\int_0^\infty e^{-\frac{t'}{\tau}+i\omega_{n1}t'}dt' = \left.\frac{e^{-\frac{t'}{\tau}+i\omega_{n1}t'}}{-\frac{1}{\tau}+i\omega_{n1}}\right|_0^\infty = \frac{1}{\frac{1}{\tau}-i\omega_{n1}} = \frac{\tau}{1-i\omega_{n1}\tau}$$

a) In this case $n=2$ $\quad \omega_{21} = \dfrac{3}{2}\dfrac{\hbar\pi^2}{ma^2}$

$$c_2^{(1)} = \frac{\lambda}{i\hbar}\frac{\tau}{1-i\omega_{21}\tau}\left(-\frac{16a}{9\pi^2}\right) = -\frac{\lambda}{i\hbar}\frac{\tau(1+i\omega_{21}\tau)}{1+\omega_{21}^2\tau^2}\frac{16a}{9\pi^2} = \frac{i16\lambda a}{9\pi^2\hbar}\frac{\tau(1+i\omega_{21}\tau)}{1+\omega_{21}^2\tau^2}$$

$$P_2^{(1)} = |c_2^{(1)}|^2 = \left(\frac{16}{9}\right)^2\frac{\lambda^2a^2\tau^2}{\pi^4\hbar^2}\frac{1}{1+\omega_{21}^2\tau}$$

c) Similarly to case a) here we have $n=4$ and $\omega_{41} = \dfrac{15}{2}\dfrac{\hbar\pi^2}{ma^2}$

$$P_4^{(1)} = \left(\frac{32}{225}\right)^2\frac{\lambda^2a^2\tau^2}{\pi^4\hbar^2}\frac{1}{1+\omega_{21}^2\tau}$$

b) In this case $P_3^{(1)} = 0$ because $\langle 3|x|1\rangle = 1$.

5. a) If $v(\vec{r})$ is weak we can use the Born approximation for scattering amplitude:

$$f_{atom}(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}'} v(\vec{r}')\,d\vec{r}' \qquad \vec{q} = \vec{k}' - \vec{k} \qquad q = 2k\sin\frac{\theta}{2}$$

$$f_{atom}(\theta,\phi) = -\frac{mg}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}' - \alpha r^2}\,d\vec{r} = -\frac{mg}{2\pi\hbar^2}\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{q_x^2}{4\alpha}}\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{q_y^2}{4\alpha}}\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{q_z^2}{4\alpha}}$$

$$= -\frac{mg\sqrt{\pi}}{2\hbar^2\alpha^{3/2}}\,e^{-\frac{q^2}{4\alpha}} = f_{atom}(\theta) \leftarrow$$ no ϕ dependence because the potential is central

Then

$$\frac{d\sigma_{atom}}{d\Omega} = |f_{atom}(\theta)|^2 = \frac{m^2g^2\pi}{4\hbar^4\alpha^3}\,e^{-\frac{q^2}{2\alpha}} \qquad q^2 = 4k^2\sin^2\frac{\theta}{2}$$

b) For the potential of all atoms

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}'} V(\vec{r}')\,d\vec{r}' = -\frac{mg}{2\pi\hbar^2}\sum_j \int e^{i\vec{q}\cdot\vec{r}'} v(\vec{r}' - \vec{R}_j)\,d\vec{r}$$

$$= -\frac{mg}{2\pi\hbar^2}\sum_j e^{i\vec{q}\cdot\vec{R}_j}\int e^{i\vec{q}\cdot\vec{r}'} v(\vec{r}')\,d\vec{r} = f_{atom}(\theta)\left[\sum_j e^{i\vec{q}\cdot\vec{R}_j}\right]$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{d\sigma_{atom}}{d\Omega}\left[\sum_j e^{i\vec{q}\cdot\vec{R}_j}\right]^2$$

6. a) The Hamiltonian of an electron in the magnetic field is $(0 \le t \le t_f)$:

$$H = \frac{e}{m}\vec{B}\cdot\vec{S} = \frac{e\hbar}{2m}\vec{B}\cdot\vec{\sigma} = \frac{e\hbar}{2m}(B_x\sigma_x + B_z\sigma_z) = \frac{e\hbar}{2m}\begin{pmatrix} B_0 - \beta t & B' \\ B' & -(B_0-\beta t)\end{pmatrix}$$

b) If $B' \ll B_0$ then at $t=0$ we essentially have

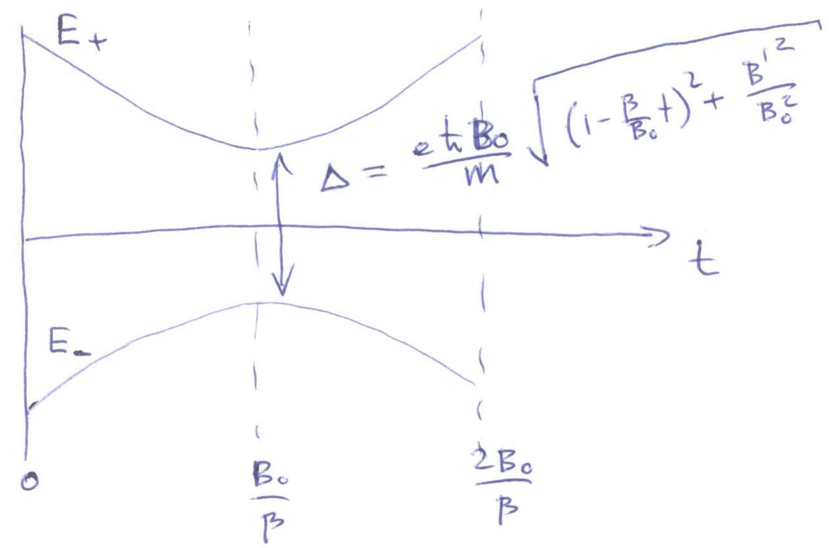
$H(0) \approx \frac{e\hbar B_0}{2m}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$ and $|\chi(0)\rangle = |\downarrow\rangle$ corresponds to the lowest of the two energies, $E_- = \frac{-e\hbar B_0}{2m}$

For any t between 0 and t_f we can find $E_\pm(t)$ - the instantaneous eigenvalues

$$H(t) = \frac{e\hbar B_0}{2m}\begin{pmatrix} 1-\frac{\beta}{B_0}t & \frac{B'}{B_0} \\ \frac{B'}{B_0} & -(1-\frac{\beta}{B_0}t)\end{pmatrix} \equiv \begin{pmatrix} \lambda & \gamma \\ \gamma & -\lambda\end{pmatrix}$$

$$(E-\lambda)(E+\lambda) - \gamma^2 = 0 \Rightarrow E^2 - \lambda^2 - \gamma^2 = 0 \Rightarrow E = \pm\sqrt{\lambda^2+\gamma^2}$$

$$E_\pm(t) = \pm\frac{e\hbar B_0}{2m}\sqrt{\left(1-\frac{\beta}{B_0}t\right)^2 + \frac{B'^2}{B_0^2}}$$



According to the adiabatic theorem, $|\chi(t)\rangle$ must remain the lowest in energy. Hence at $t \ge t_f$

$|\chi(t)\rangle \approx |\uparrow\rangle$ because $H(t \ge t_f) \approx -\frac{\beta}{B_0}H(0)$

However, the transitions between E_+ and E_- will not occur only if E_+ and E_- are never become degenerate. The gap between E_+ and E_- (let us call it Δ) is lowest at $t = \frac{B_0}{\beta}$. It vanishes completely when $B' = 0$. Therefore the adiabatic theorem is applicable only if $B' \neq 0$. Otherwise there is no way to predict the final state.

c) The adiabatic theorem says the transitions to other state(s) vanish when $T_e \gg T_i$, where T_e is the characteristic time for changes in the Hamiltonian and T_i is the characteristic time for changes in the wave function.

In our case $\quad T_e \approx \frac{B_0}{\beta} \leftarrow$ the time which the system spends in the dangerous region where $B' \approx |B_0 - \beta t|$

$$T_i \approx \frac{\hbar}{E_+ - E_-} = \frac{\hbar}{\Delta} < \frac{m}{eB'}$$

which leads to the condition:

$$\frac{eB'^2}{m\beta} \gg 1 \qquad \text{or} \qquad \frac{m\beta}{eB'^2} \ll 1$$