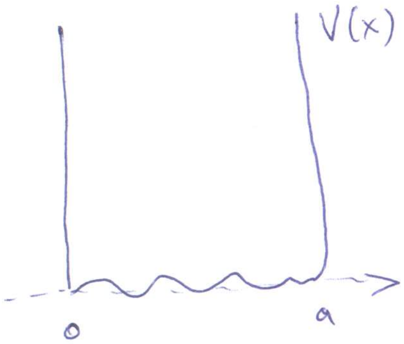
Bohr-Sommerfeld quantization rules

In the previous lecture we have obtained that up to first order in $\hbar$ the semiclassical approximation takes the following form

$$\psi \approx \frac{1}{\sqrt{p(x)}}\left[C_+ e^{\frac{i}{\hbar}\int p(x)dx} + C_- e^{-\frac{i}{\hbar}\int p(x)dx}\right]$$

where $p(x) = \sqrt{2m(E-V(x))}$

Now let us apply this formula to the case of a potential with two infinite (vertical) walls. We can rewrite $\psi(x)$ as



$$\psi(x) = \frac{1}{\sqrt{p(x)}}\left[C\sin\phi(x) + D\cos\phi(x)\right]$$

with $\phi(x) = \frac{1}{\hbar}\int_0^x p(x')\,dx'$

Since the potential becomes infinite at $x=0$ the boundary condition is $\psi(0)=0$. Now since $\phi(0)=0$ D must be equal to zero. Also $\psi(x)$ goes to zero at $x=a$. Then

$$\phi(a) = n\pi \qquad n=1,2,3,\dots$$

($n=0$ excluded, because in that case $\psi=0$)

Hence we conclude that

$$\int_0^a p(x)\,dx = n\pi\hbar$$

$\longrightarrow$ this condition determines energies

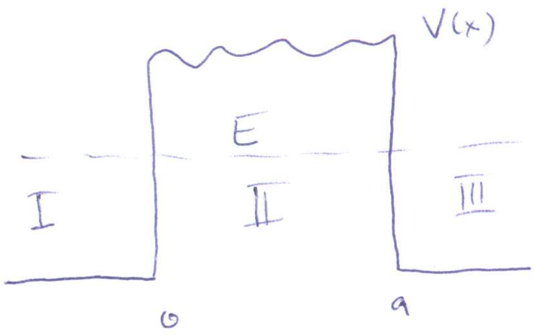
If we consider the infinite square well (which we know how to solve exactly) then

$$n\pi\hbar = \int_0^a p(x)dx = \int_0^a \sqrt{2mE}\,dx = \sqrt{2mE}\,a \implies E = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

In this particular case the WKB approximation recovers the exact formula for E_n.

Barrier tunneling in WKB approximation

We can also consider scattering (i.e. when $E < V$) and transmission through a potential barrier.



In region I:

$$\psi = Ae^{ikx} + Be^{-ikx} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

In region III

$$\psi = Fe^{ikx}$$

As we know the transmission coefficient is defined as

$$T = \frac{|F|^2}{|A|^2}$$

In region II we can use the WKB approximation

$$\psi = \frac{C}{\sqrt{\varkappa(x)}} e^{\int_0^x \varkappa(x')dx'} + \frac{D}{\sqrt{\varkappa(x)}} e^{-\int_0^x \varkappa(x')dx'}$$

where $\varkappa(x) = \frac{1}{\hbar}\sqrt{2m(V(x)-E)} = \frac{|p(x)|}{\hbar}$

If we assume that the barrier is wide and high so that the tunneling probability is small then the term $\frac{C}{\sqrt{\varkappa(x)}} e^{\int_0^x \varkappa(x')dx'}$ must also be small (because ψ must decay). The relative amplitudes of the incident and transmitted waves are determined by the total decrease of the exponential over the nonclassical region

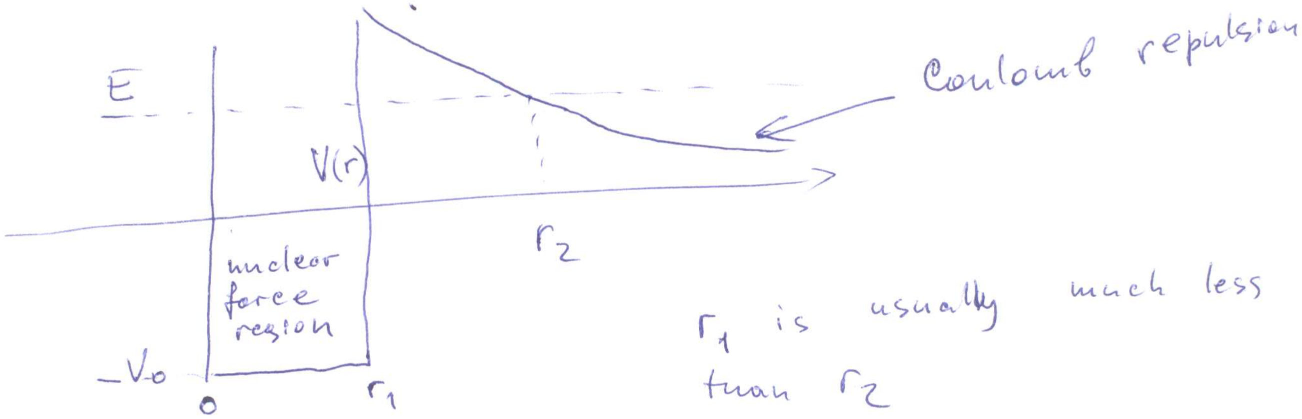
$$\frac{|F|}{|A|} \sim e^{-\int_0^a \varkappa(x')dx}$$

and

$$T = e^{-2\gamma} \qquad \gamma = \int_0^a \varkappa(x')dx$$

Gamow's theory of alpha decay

The WKB approximation in the context of barrier tunneling can be used to describe the spontaneous emission of α-particles by some radioactive nuclei.



r_1 is usually much less than r_2

Turning point are r_1 and r_2

$$\underbrace{\frac{1}{4\pi\varepsilon_0}\frac{2Ze^2}{r_2}}_{\substack{\text{Coulomb repulsion of} \\ \text{a positively charged } \alpha\text{-particle} \\ \text{from the rest of the nucleus} \\ \text{at large distances}}} = E$$

(the charge of the alpha particle is $2e$)

$$\gamma = \int_{r_1}^{r_2} \kappa(r)\,dr = \frac{\sqrt{2m}}{\hbar}\int_{r_1}^{r_2}\sqrt{V(r)-E}\,dr = \frac{\sqrt{2m}}{\hbar}\int_{r_1}^{r_2}\sqrt{\frac{2Ze^2}{4\pi\varepsilon_0 r}-E}\,dr$$

$$= \frac{\sqrt{2mE}}{\hbar}\int_{r_1}^{r_2}\sqrt{\frac{r_2}{r}-1}\,dr$$

Here we use the integral taken from a table of integrals

$$\int\sqrt{\frac{a}{x}-1}\,dx = \sqrt{\frac{a}{x}-1}\,x - \frac{a}{2}\arctan\left[\frac{\sqrt{\frac{a}{x}-1}}{2}\,\frac{2x-a}{x-a}\right]$$

So

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\left(\sqrt{\frac{r_2}{r}-1}\,r - \frac{r_2}{2}\arctan\left[\frac{\sqrt{\frac{r_2}{r}-1}}{2}\,\frac{2r-r_2}{r-r_2}\right]\right)\Bigg|_{r_1}^{r_2} =$$

$$= \frac{\sqrt{2mE}}{\hbar}\left(-\sqrt{(r_2-r_1)r_1} + \frac{r_2}{2}\arctan\left[\frac{1}{2}\,\frac{r_2-2r_1}{\sqrt{(r_2-r_1)r_1}}\right] + \frac{\pi r_2}{4}\right)$$

Note that $\arctan(x) \underset{x\to\infty}{\approx} \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} + \dots$

If $r_2 \gg r_1$ then

$$\gamma = \frac{\sqrt{2mE}}{\hbar}\left(-\sqrt{r_2 r_1} + \frac{r_2}{2}\left(\frac{\pi}{2} - 2\sqrt{\frac{r_1}{r_2}}\right) + \frac{\pi r_2}{4}\right) =$$

$$= \frac{\sqrt{2mE}}{\hbar}\left[\frac{\pi}{2} r_2 - 2\sqrt{r_1 r_2}\right]$$

It can also be rewritten as

$$\gamma = k_1 \frac{Z}{\sqrt{E}} - k_2 \sqrt{Z r_1}$$

$$k_1 = \left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{\pi\sqrt{2m}}{\hbar} \approx 1.98 \text{ MeV}^{1/2}$$

$$k_2 = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^{1/2}\frac{4\sqrt{m}}{\hbar} \approx 1.485 \text{ fm}^{-1/2}$$

The frequency of the α-particle collision with the barrier is given by its speed over twice the nuclear radius, $\frac{v}{2r_1}$. The probability of emission per unit time is $\frac{v}{2r_1} e^{-2\gamma}$

The lifetime of the nucleus is

$$\tau = \frac{2r_1}{v} e^{2\gamma}$$