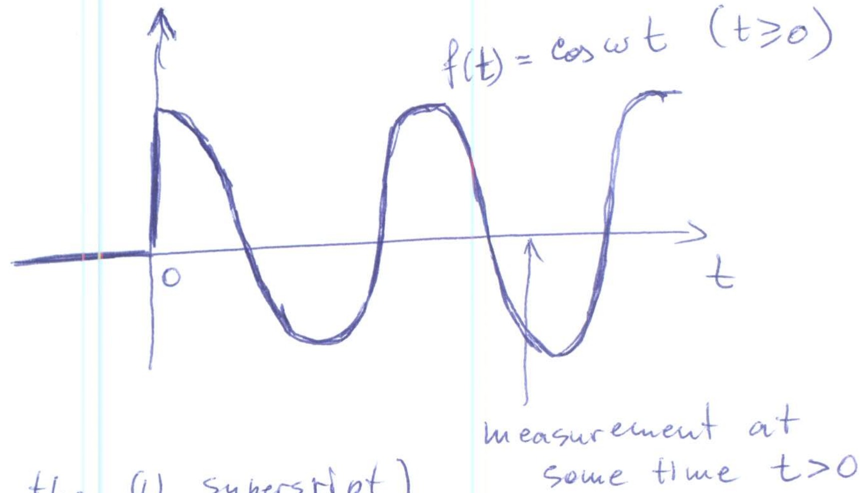
Harmonic perturbation

As an application of time-dependent perturbation theory let us consider a perturbation that is turned on at $t=0$ and is subsequently monochromatically harmonic in time. The perturbation acts on a system whose Hamiltonian is H^0. Such a model is relevant to describe an atom that interacts with a (weak) electromagnetic field. The explicit form of the perturbation is

$$H'(\vec{r},t) = \begin{cases} 0 & , \; t<0 \\ \mathcal{H}'(\vec{r}) \cos \omega t & , \; t \geq 0 \end{cases}$$



Using the result of the previous lecture, namely

$$c^{(1)}_{fi}(t) = \frac{\mathcal{H}'_{fi}}{i\hbar} \int_{t_0}^{t} e^{i\omega_{fi}t'} f(t')\, dt'$$

we get (we will drop the (1) superscript)

$$c_{fi}(t) = \frac{\mathcal{H}'_{fi}}{i\hbar} \int_0^t e^{i\omega_{fi}t'} \left(\frac{e^{-i\omega t'} + e^{i\omega t'}}{2} \right) dt' =$$

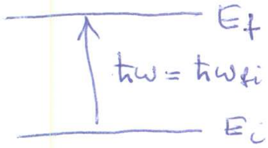
$$= -\frac{\mathcal{H}'_{fi}}{2\hbar} \left[\frac{e^{i(\omega_{fi}-\omega)t} - 1}{\omega_{fi} - \omega} + \frac{e^{i(\omega_{fi}+\omega)t} - 1}{\omega_{fi} - \omega} \right]$$

If we use the relation $e^{i\alpha} - 1 = 2i e^{\frac{i\alpha}{2}} \sin \frac{\alpha}{2}$ we can rewrite it as

$$c_{fi}(t) = -\frac{i\mathcal{H}'_{fi}}{\hbar} \left[\frac{e^{i(\omega_{fi}-\omega)t/2} \sin \frac{(\omega_{fi}-\omega)t}{2}}{\omega_{fi} - \omega} + \frac{e^{i(\omega_{fi}-\omega)t/2} \sin \frac{(\omega_{fi}+\omega)t}{2}}{\omega_{fi} + \omega} \right]$$

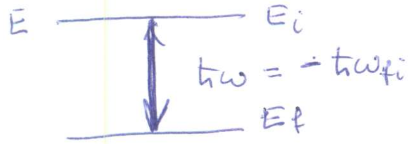
c_{fi} is big when $\omega \simeq \pm \omega_{fi}$ (resonant frequency)

In the case $\omega \simeq +\omega_{fi}$ $\quad E_f > E_i$ ← system absorbs energy and "jumps" to a higher energy level $E_f = E_i + \hbar\omega$



Resonant absorption

In the case $\omega \simeq -\omega_{fi}$ $\quad E_f < E_i$ ← perturbation induces a decay in energy $E_f = E_i - \hbar\omega$

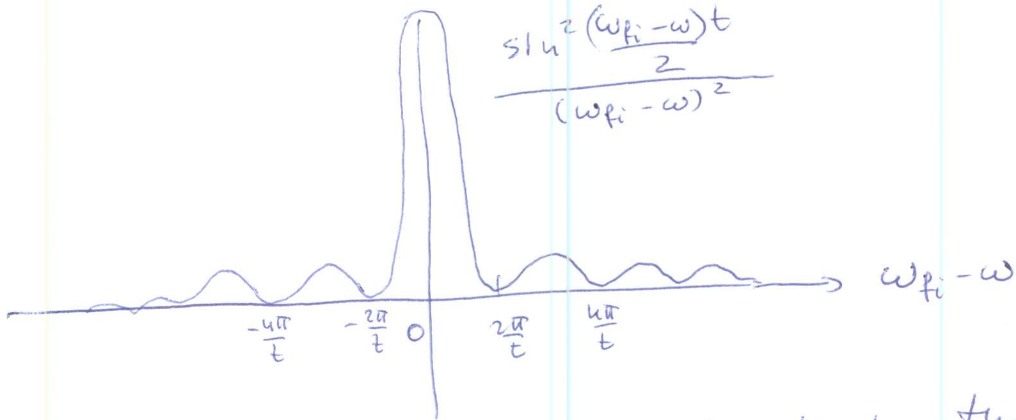


Stimulated emission

The decay process is stimulated by a photon of the same frequency in the perturbation field (stimulated emission)

Let us consider the case $\omega_{fi} > 0$. Under this conditions the first term dominates and the probability of transition is

$$P_{fi} = |C_{fi}|^2 = \frac{|H'_{fi}|^2}{\hbar^2(\omega_{fi}-\omega)^2} \sin^2\left[\frac{(\omega_{fi}-\omega)t}{2}\right]$$



From this scetch it is evident that the states falling in the interval $|\hbar\omega_{fi} - \hbar\omega| = |E_f - (E_i + \hbar\omega)| \leq \frac{2\pi\hbar}{t} \simeq \Delta E$ have the greatest probability of being excited, after the perturbation has acted for t seconds. Hence, the above inequatity provides the uncertainty of the energies (ater time t) that will be observed:

$$\Delta E \simeq \frac{\hbar}{t} \qquad E_f \simeq E_i + \hbar\omega \pm \Delta E$$

Our analysis has returned the conservation of energy modified by the uncertainty relation

Let us now consider long-time evolution. The expression for P_{fi} in the limit $t \to \infty$ (or, alternatively, $\omega \to \infty$) becomes

$$P_{fi} \to \frac{\pi t}{2\hbar^2} |\mathcal{H}'_{fi}|^2 \delta(\omega_{fi} \mp \omega)$$

because

$$\delta(\omega) = \frac{2}{\pi} \lim_{t \to \infty} \frac{\sin^2 \frac{\omega t}{2}}{t\omega^2}$$

The corresponding transition probability rate is

$$w_{fi} = \frac{\pi}{2\hbar^2} |\mathcal{H}'_{fi}|^2 \delta(\omega_{fi} \mp \omega)$$

In the above formulae for w_{fi} and P_{fi} the presence of the δ-function constitutes the fact that in the long-time limit the Fourier transform of the perturbation becomes sharply peaked around the frequency of perturbation. Thus, the system will only see a single frequency. Since the uncertainty in energy $\frac{\hbar}{t}$ vanishes in this limit, the argument of the delta-function is also an expression for the conservation of energy.

In the short-time approximation, $(\omega_{fi} - \omega)t \ll 1$, the expression for P_{fi} may be expanded into a Taylor series in t:

$$P_{fi} = \frac{t^2 |\mathcal{H}'_{fi}|^2}{4\hbar^2} \quad \text{and then} \quad w_{fi} = \frac{t |\mathcal{H}'_{fi}|^2}{4\hbar^2}$$

At early times the rate at which transitions to the f-th state occur grows linearly with time.