Berry's phase

Let us investigate how the final state differs from the initial state if the parameters in the Hamiltonian are carried adiabatically around some closed cycle.

In the previous lecture we found out that a particle which starts in the n-th state of $H(t=0)$ remains in the n-th state of $H(t)$, picking up only a phase factor:

$$\Psi_n = e^{i[\theta_n(t)+\gamma_n(t)]}\,\psi_n(t)$$

where

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')\,dt' \quad \text{– dynamic phase}$$

and

$$\gamma_n(t) = i\int_0^t \left\langle \psi_n(t') \middle| \frac{\partial}{\partial t'}\psi_n(t')\right\rangle dt' \quad \text{– geometric phase}$$

Now suppose $\psi_n(t)$ depends on t through some parameter $R(t)$. Thus,

$$\frac{\partial \psi_n}{\partial t} = \frac{\partial \psi_n}{\partial R}\frac{dR}{dt}$$

and

$$\gamma_n(t) = i\int_0^t \left\langle \psi_n \middle| \frac{\partial \psi_n}{\partial R}\right\rangle \frac{dR}{dt'}\,dt' = i\int_{R_i}^{R_f}\left\langle \psi_n \middle| \frac{\partial \psi_n}{\partial R}\right\rangle dR$$

where R_i and R_f are the initial and final values of $R(t)$. If $R_f(T)=R_i(0)$ then $\gamma_n(T)=0$

However, if we assume that there is more than one parameter (say N of them) then

$$\frac{\partial \varphi_n}{\partial t} = \frac{\partial \varphi_n}{\partial R_1}\frac{dR_1}{dt} + \frac{\partial \varphi_n}{\partial R_2}\frac{dR_2}{dt} + \dots + \frac{\partial \varphi_n}{\partial R_N}\frac{dR_N}{dt} = (\nabla_R \varphi_n)\cdot \frac{d\vec{R}}{dt}$$

$$\vec{R} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{pmatrix} \qquad \nabla_R = \begin{pmatrix} \frac{\partial}{\partial R_1} \\ \frac{\partial}{\partial R_2} \\ \vdots \\ \frac{\partial}{\partial R_N} \end{pmatrix}$$

and

$$\gamma_n(t) = i\int_{\vec{R}_i}^{\vec{R}_f} \langle \varphi_n | \nabla_R \varphi_n \rangle \cdot d\vec{R}$$

(here $d\vec{R}$ stands for a vector)

If the Hamiltonian returns to its original form after a time T, the net geometric phase change is

$$\gamma_n(T) = i\oint \langle \varphi_n | \nabla_R \varphi_n \rangle \cdot d\vec{R}$$

It should be noted that $\gamma_n(T)$ depends on the path only (in the space of parameters R_i), not on how fast the path is traversed. In contrast

$$\theta_n(T) = -\frac{1}{\hbar}\int_0^T E_n(t')\,dt'$$

Despite γ_n being just a phase factor it may be measurable. Suppose a beam of particles (all in state Ψ) is split in two and then one of them is passed through an adiabatically changed potential. Then when the two beams are recombined, the total wave function has the form

$$\Psi = \frac{1}{2}\Psi_0 + \frac{1}{2}\Psi_0 e^{i\Gamma}$$

whe Γ is the extra phase (in part dynamic and in part geometric)

$$|\Psi|^2 = \frac{1}{2}|\Psi_0|^2 (1+e^{i\Gamma})(1+e^{-i\Gamma}) = \frac{1}{2}|\Psi_0|^2 (1+\cos\Gamma) =$$

$$= |\Psi_0|^2 \cos^2\frac{\Gamma}{2}$$

When the parameter space is 3-dimensional, $\vec{R} = (R_1, R_2, R_3)$, Berry's formula looks similar to the expression for magnetic flux

$$\Phi = \int_S \vec{B}\cdot d\vec{s}$$

If we write the magnetic field as $\vec{B} = (\nabla\times\vec{A})$ and use the Stoke's theorem

$$\Phi = \int_S (\nabla\times\vec{A})\cdot d\vec{s} = \oint_C \vec{A}\cdot d\vec{\ell}$$

Thus Berry's phase can be thought of as the "flux" of a "magnetic field"

$$\vec{B} = i\nabla_R \times \langle\varphi_n|\nabla_R \varphi_n\rangle$$

through the closed loop trajectory in parameter space.

$$\gamma_n(T) = i\int_S [\nabla_R \times \langle\varphi_n|\nabla_R\varphi_n\rangle]\cdot d\vec{s}$$