The Aharonov - Bohm effect

The Aharonov-Bohm effect has great conceptual importance because it allows to test whether potentials are physical or just a convenient tool for calculating force fields.

In classical electrodynamics the potentials φ and $\vec{A}$ are not directly measurable — the physical quantities are the electric and magnetic fields

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

The fundamental laws, as written in their usual forms (Maxwell equations and the Lorentz force rule) make no reference to potentials. The latter are just convenient constructs and are not defined uniquely:

$$\varphi \to \varphi' = \varphi - \frac{\partial \Lambda}{\partial t} \qquad \vec{A} \to \vec{A}' = A + \nabla\Lambda$$

In quantum mechanics the potentials are more important as they appear in the Hamiltonian

$$H = \frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2 + q\varphi$$

Nevertheless the theory is still invariant under the gauge transformatios. For a long time it was taken for granted that there could be no electromagnetic influences in regions where $\vec{E}$ and $\vec{B}$ are zero. Howeverr in 1959 Aharonov and Bohm (and previously Ehrenberg and Siday in 1949) showed that the vector potential can affect the quantum behavior of a charged particle, even when it is moving through a region in which the field itself is zero.

Let us consider a particle constrained to move in a circle of radius b. Along the axis runs a solenoid of radius $a<b$ carrying a steady electric current I. Adopting the gauge $\nabla\cdot\vec{A}=0$ (so called Coulomb gauge) we have for the vector potential:

$$\vec{A} = \frac{\Phi}{2\pi r}\hat{\phi} \quad (r>a)$$

where $\Phi = \pi a^2 B$ is the magnetic flux through the solenoid

and

$$\varphi = 0 \quad \text{(solenoid is uncharged)}$$

Then the Hamiltonian becomes

$$H = \frac{1}{2m}\left[-\hbar^2\nabla^2 + q^2A^2 + 2i\hbar q\vec{A}\cdot\nabla\right]$$

Since the wave function depends only on the azimutal angle $\nabla \to \frac{\hat{\phi}}{b}\frac{d}{d\phi}$

(recall that in spherical coordinates

$$\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$)

The Schrödinger equation then reads

$$\frac{1}{2m}\left[-\frac{\hbar^2}{b^2}\frac{d^2}{d\phi^2} + \left(\frac{q\Phi}{2\pi b}\right)^2 + i\frac{\hbar q\Phi}{\pi b^2}\frac{d}{d\phi}\right]\psi(\phi) = E\psi(\phi)$$

or

$$\frac{d^2\psi}{d\phi^2} - 2i\beta\frac{d\psi}{d\phi} + \epsilon\psi = 0 \qquad \beta = \frac{q\Phi}{2\pi\hbar} \qquad \epsilon = \frac{2mb^2E}{\hbar^2} - \beta^2$$

The solutions of the latter equation are

$$\psi = C e^{i\lambda\phi} \quad \text{with } \lambda = \beta \pm \sqrt{\beta^2 + \epsilon} = \beta \pm \frac{b}{\hbar}\sqrt{2mE}$$

Continuity at $\phi = 2\pi$ requires that λ be an integer

$$\beta \pm \frac{b}{\hbar}\sqrt{2mE} = n$$

Then

$$E_n = \frac{\hbar^2}{2mb^2}\left(n - \frac{q\Phi}{2\pi\hbar}\right)^2 \qquad n = 0, \pm 1, \pm 2, \ldots$$

The solenoid lifts the two-fold degeneracy of the bead-on-a-ring. What is more important is that the allowed energies depend on the field inside the solenoid, even though the field at the location of the particle is zero.

More generally, suppose a particle is moving through a region where $\vec{B}$ is zero (so $\nabla \times \vec{A} = 0$) but $\vec{A}$ is not. The TDSE

$$\left[\frac{1}{2m}\left(-i\hbar\nabla - q\vec{A}\right)^2 + V\right]\psi = i\hbar\frac{\partial\psi}{\partial t}$$

can be simplified by writing $\psi = e^{ig}\psi'$

where $g(\vec{r}) = \frac{q}{\hbar}\int_{\vec{r}_{ref}}^{\vec{r}} A(\vec{r}\,')\cdot d\vec{r}\,'$

and $\vec{r}_{ref}$ is some arbitrary chosen reference point.

In terms of ψ', the gradient of ψ is

$$\nabla\psi = e^{ig}(i\nabla g)\psi' + e^{ig}(\nabla\psi')$$

(we assume $A \neq A(t)$ here, i.e. we deal with static $\vec{A}$)

but $\nabla g = \frac{q}{\hbar}\vec{A}$, so

$$\left(-i\hbar\nabla - q\vec{A}\right)\psi = -i\hbar e^{ig}\nabla\psi'$$

and

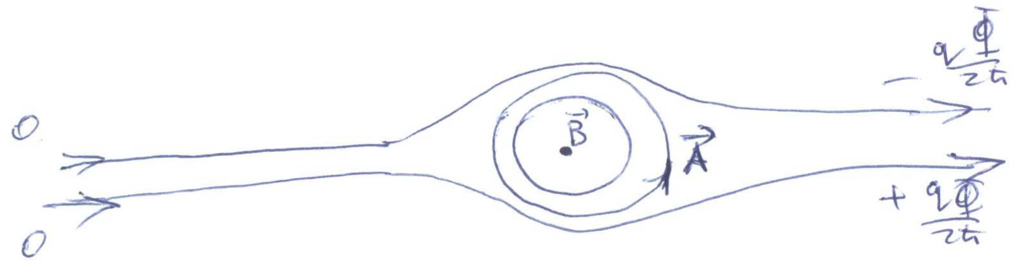
$$\left(-i\hbar\nabla - q\vec{A}\right)^2\psi = -\hbar^2 e^{ig}\nabla^2\psi'$$

Putting this back to the TDSE and cancelling e^{ig} gives

$$-\frac{\hbar^2}{2m}\nabla^2\psi' + V\psi' = i\hbar\frac{\partial \psi'}{\partial t}$$

Evidently ψ' satisfies the TDSE without $\vec{A}$ and again recall that $\psi = e^{ig}\psi'$

The classical experiment that confirms the Aharonov-Bohm effect involves a beam of electrons that is split into two and passes on both sides of a solenoid (but only in the region where $\vec{B}=0$). But $\vec{A}$ is not zero where the electrons pass. As a result the two beams arrive with different phases



$$g = \frac{q}{\hbar}\int \vec{A}\cdot d\vec{r} = \frac{q\Phi}{2\pi\hbar}\int\left(\frac{1}{r}\hat{\phi}\right)\cdot\left(r\hat{\phi}\,d\phi\right) = \pm\frac{q\Phi}{2\hbar}$$

$\Delta g = \frac{q\Phi}{\hbar} \leftarrow$ can be measured : $|\psi|^2 = |\psi_0|\cos^2\frac{g}{2}$

Aharonov - Bohm effect can be regarded as an example of geometric phase

Vector potential of an infinite solenoid

$$\oint \vec{A}\cdot d\vec{l} = \int (\nabla\times\vec{A})\cdot d\vec{s} = \int \vec{B}\cdot d\vec{s} = \Phi$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl}$$ ← Ampere's law

The latter ($\oint \vec{B}\cdot d\vec{l}$) is similar to $\oint \vec{A}\cdot d\vec{l}$ (just need to replace $\vec{B}\rightarrow\vec{A}$ and $\mu_0 I_{encl}\rightarrow\Phi$)

For a loop outside the solenoid the flux is

$$\int \vec{B}\cdot d\vec{s} = \mu_0 n I\,\underbrace{(\pi a^2)}_{\text{since the field only extends out to } a}$$

$$\vec{A} = \frac{\mu_0 n I}{2}\frac{a^2}{r} \qquad r>a$$

n turs per unit length