1. a) This trial wave function will yield an upper bound to the first excited energy level. This is because the potential is symmetric with respect to the point $x = \frac{a}{2}$ and the ground state wave function is therefore symmetric with respect to that point. Hence, our trial wave function is automatically orthogonal to the ground state wave function. In other words, in the expansion

$$\psi_{trial} = \sum_{i=1}^{\infty} c_i \psi_i \qquad \text{where } \psi_i \text{ are exact eigenstates}$$

we have $c_1 = 0$ because of the orthogonality

then

$$E_{trial} = \langle \psi_{trial} | H | \psi_{trial} \rangle = \langle \sum_{i=1}^{\infty} c_i \psi_i | H | \sum_{j=1}^{\infty} c_j \psi_j \rangle =$$

$$= \sum_{i=2}^{\infty} |c_i|^2 E_i \geq E_2 \sum_{i=2}^{\infty} |c_i|^2 = E_2$$

b) The trial wave function is

$$\psi_{trial} = \begin{cases} Ax , & 0 \leq x \leq \frac{a}{4} \\ A(\frac{a}{2} - x) , & \frac{a}{4} \leq x \leq \frac{3a}{4} \\ A(x-a) , & \frac{3a}{4} \leq x \leq a \end{cases}$$

$$1 = \langle \psi_{trial} | \psi_{trial} \rangle = 4A^2 \int_0^{a/4} x^2 dx = 4A^2 \frac{x^3}{3} \Big|_0^{a/4} = \frac{A^2 a^3}{48} \Rightarrow A^2 = \frac{48}{a^3}$$

$$E_{trial} = \langle \psi_{trial} | H | \psi_{trial} \rangle = \int_0^a \psi_{trial} \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_{trial} \right] dx =$$

$$= -\frac{\hbar^2}{2m} \int_0^a \psi_{trial} \left[A\delta(x) - 2A\delta(x - \tfrac{a}{4}) + 2A\delta(x - \tfrac{3a}{4}) - A\delta(x-a) \right] dx =$$

$$= -\frac{\hbar^2}{2m} \left[0 - 2A \cdot A\frac{a}{4} + 2A(-A\frac{a}{4}) - 0 \right] = \frac{A^2 \hbar^2 a}{2m} = \frac{48 \hbar^2}{2ma^2} = \frac{24\hbar^2}{ma^2}$$

$$E_2^{exact} = \frac{2^2 \pi^2 \hbar^2}{2ma^2} = \frac{2\pi^2\hbar^2}{ma^2}$$

$$\frac{E_{trial}}{E_2^{exact}} = \frac{12}{\pi^2} \approx 1.216$$

2. First order correction:

$$E_n^{(1)} = H'_{nn} = \underbrace{\int \psi_n^* \left(-\gamma i \hbar \frac{d}{dx} \psi_n\right)}_{\text{odd}} dx = 0$$

Second order correction:

$$E_n^{(2)} = \sum_{k \neq n} \frac{|H'_{nk}|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$H'_{nn} = \gamma \langle n | \hat{p} | k \rangle =$$

$$= i\gamma \sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{k}\, \delta_{n,k-1} + \sqrt{n}\, \delta_{k,n-1} \right)$$

$$E_n^{(0)} = \hbar\omega \left(n + \frac{1}{2}\right)$$

Hence

$$E_0^{(2)} = \frac{\gamma^2 \frac{m\hbar\omega}{2}}{\frac{\hbar\omega}{2} - \frac{3}{2}\hbar\omega} = -\frac{\gamma^2 m}{2}$$

3. $$E_{gr}^{(0)} = \frac{\pi^2\hbar^2}{2m_1a^2} + \frac{\pi^2\hbar^2}{2m_2a^2}$$

$$E_{exc}^{(0)} = \frac{\pi^2\hbar^2}{2m_1a^2} + \frac{4\pi^2\hbar^2}{2m_2a^2}$$ (assuming $m_2 > m_1$, i.e. the second particle is heavier)

$$\psi_{gr}^{(0)} = \frac{2}{a}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a} \qquad \psi_{exc}^{(0)} = \frac{2}{a}\sin\frac{\pi x_1}{a}\sin\frac{2\pi x_2}{a}$$

If there were a weak perturbation $V = \lambda\delta(x_1 - x_2)$ then the first order corrections to the energies would be

$$E_{gr}^{(1)} = \int_0^a\int_0^a \psi_{gr}^{(0)} V \psi_{gr}^{(0)}\, dx_1\, dx_2 = \frac{4}{a^2}\lambda\int_0^a\int_0^a \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}\,\delta(x_1-x_2)\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}\,dx_1\,dx_2 =$$

$$= \frac{4\lambda}{a^2}\int_0^a \sin^4\frac{\pi x_1}{a}\,dx_1 = \frac{4\lambda}{a^2}\,\frac{a}{\pi}\int_0^\pi \sin^4 y\,dy = \frac{4\lambda}{a\pi}\int_0^\pi \sin^4 y\,dy =$$

$$= \frac{4\lambda}{a\pi}\int_0^\pi \frac{1}{4}(1-\cos 2y)(1-\cos 2y)\,dy = \frac{\lambda}{a\pi}\int_0^\pi (1 - 2\cos 2y + \cos^2 2y)\,dy =$$

$$= \frac{\lambda}{a\pi}\left[\pi + \int_0^\pi \frac{1}{2}(1+\cos 4y)\,dy\right] = \frac{\lambda}{a\pi}\cdot\frac{3\pi}{2} = \frac{3\lambda}{2a}$$

$$E_{exc}^{(1)} = \int_0^a\int_0^a \psi_{exc}^{(0)} V \psi_{exc}^{(0)}\,dx_1\,dx_2 = \frac{4}{a^2}\lambda\int_0^a\int_0^a \sin\frac{\pi x_1}{a}\sin\frac{2\pi x_2}{a}\,\delta(x_1-x_2)\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}\,dx_1\,dx_2 =$$

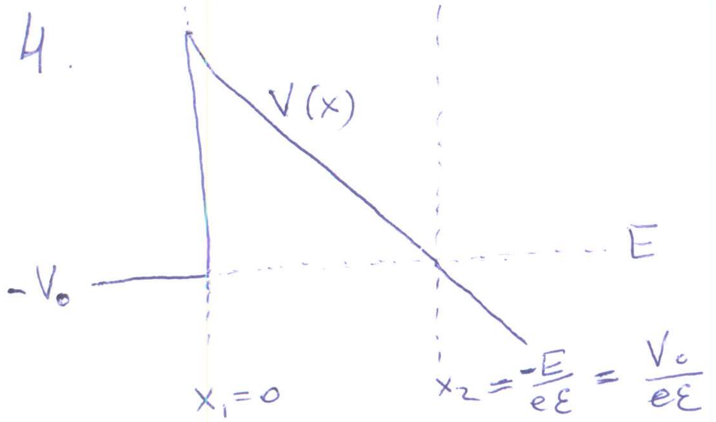
$$= \frac{4\lambda}{a\pi}\int_0^\pi \sin^2 y\,\sin^2 2y\,dy = \frac{4\lambda}{a\pi}\int_0^\pi \frac{1}{4}(1-\cos 2y)(1-\cos 4y)\,dy =$$

$$= \frac{\lambda}{a\pi}\int_0^\pi \Big(1 - \cos 2y - \cos 4y + \underbrace{\cos 2y\cos 4y}_{\frac{1}{2}[\cos 2y + \cos 6y]}\Big)\,dy = \frac{\lambda}{a}$$

Change in transition frequency:

$$\Delta\nu = \frac{E_{exc} - E_{gr}}{h} - \frac{E_{exc}^{(0)} - E_{gr}^{(0)}}{h} \approx \frac{E_{exc}^{(1)} - E_{exc}^{(1)}}{h} = \frac{\frac{\lambda}{a} - \frac{3\lambda}{2a}}{h} = -\frac{1}{2}\frac{\lambda}{ah}$$

4.



$$V(x) = \begin{cases} -V_0 & , x < 0 \\ -e\mathcal{E}x & , x \geq 0 \end{cases}$$

Tunneling amplitude in WKB approximation:

$$\exp\left[-\frac{1}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V(x)-E)}\,dx\right]$$

The integral is

$$\int_{x_1}^{x_2}\sqrt{2m(V(x)-E)}\,dx = \sqrt{2m}\int_0^{\frac{|E|}{e\mathcal{E}}}\sqrt{-E-e\mathcal{E}x}\,dx = \sqrt{2m}\,|E|^{1/2}\int_0^{\frac{|E|}{e\mathcal{E}}}\left[1-\frac{e\mathcal{E}x}{|E|}\right]^{1/2}dx$$

$$= \sqrt{2m}\,|E|^{1/2}\left(\frac{|E|}{e\mathcal{E}}\right)\underbrace{\int_0^1[1-y]^{1/2}\,dy}_{2/3} = \frac{2}{3}\frac{\sqrt{2m}\,|E|^{\frac{3}{2}}}{e\mathcal{E}}$$

The tunneling probability is the square of the tunneling amplitude:

$$T = \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V(x)-E)}\,dx\right] = \exp\left[-\frac{4\sqrt{2m}\,|E|^{3/2}}{3e\mathcal{E}\hbar}\right]$$

In the ground state $E = -V_0$ so

$$T = \exp\left[-\frac{4\sqrt{2m}\,V_0^{3/2}}{3e\mathcal{E}\hbar}\right]$$