1. We can write the Hamiltonian as $H = H^0 + H'$, where $H^0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2}$ is the harmonic oscillator Hamiltonian and $H' = \frac{m\omega^2 x^2}{2}\gamma \cos 2\omega t$ is a time-dependent perturbation. The time-dependent wave function can be represented as

$$\psi(x,t) = \sum_n{}' c_n(t)\varphi_n(x)\, e^{-\frac{iE_n^{(0)}t}{\hbar}}$$

where $E_n^{(0)} = \hbar\omega(n+\frac{1}{2})$ are eigenenergies and $\varphi_n(x)$ are eigenfunctions of the harmonic oscillator. The first-order perturbation theory gives the following expressions for the expansion coefficients $c_n(t)$:

$$c_n^{(1)}(t) = \frac{1}{i\hbar}\int_0^t H'_{nm}(t')\, e^{i\omega_{nm}t'}dt'$$

In our case $m=0$ (the initial state is $|0\rangle$) and $\omega_{n0} = n\omega$.

$$c_n^{(1)} = \frac{1}{i\hbar}\int_0^t \langle n|\frac{m\omega^2 x^2\gamma}{2}\cos 2\omega t'|0\rangle\, e^{in\omega t'}dt' = \frac{m\omega^2\gamma}{2i\hbar}\langle n|x^2|0\rangle\int_0^t \cos 2\omega t'\, e^{in\omega t'}dt' =$$

$$= \frac{m\omega^2\gamma}{2i\hbar}\left[\frac{\hbar}{2m\omega}(\sqrt{2}\,\delta_{n2} + \delta_{n0})\right]\int_0^t \cos 2\omega t'\, e^{in\omega t'}dt' = -\frac{i\omega\gamma}{4}(\sqrt{2}\,\delta_{n2}+\delta_{n0})\int_0^t \cos 2\omega t'\, e^{in\omega t'}dt'$$

When $n=0$ the last integral is:

$$\int_0^t \cos 2\omega t'\, dt' = \frac{\sin 2\omega t}{2\omega}$$

When $n=2$ the integral is

$$\int_0^t \cos 2\omega t'\, e^{2i\omega t'}dt' = \int_0^t \frac{e^{2i\omega t'} + e^{-2i\omega t'}}{2} e^{2i\omega t'}dt' = \frac{1}{2}\int_0^t (e^{4i\omega t'}+1)\,dt' =$$

$$= \frac{1}{2}\,\frac{e^{4i\omega t}-1}{4i\omega} + \frac{1}{2}t$$

With that we can write the final expression for $c_n^{(1)}(t)$:

$$c_n^{(1)}(t) = \begin{cases} -i\frac{\gamma}{2}\sin 2\omega t & n=0 \\ -\frac{i\omega\gamma}{4\sqrt{2}}\left(\frac{e^{4i\omega t}-1}{4\omega i} + t\right) = -\frac{\gamma}{16\sqrt{2}}(e^{4i\omega t}-1) + i\frac{\gamma}{4\sqrt{2}}\omega t = \frac{\gamma}{16\sqrt{2}}(1-\cos 4\omega t) + i\frac{\gamma}{4\sqrt{2}}(\omega t - \frac{1}{4}\sin 4\omega t) & n=2 \\ 0, & \text{otherwise} \end{cases}$$

a) Based on the fact that

$$\langle n|x^2|k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\delta_{nk}\right)$$

it is easy to see that odd states $(k=1,3,5\ldots)$ will not be excited in any order of the perturbation theory. Thus,

$$P_{0\to 1}(t) = 0$$

b)

$$P^{(1)}_{0\to 2}(t) = |C_2^{(1)}(t)|^2 = \frac{\gamma^2}{32}\left(\frac{1}{16}(1-\cos 4\omega t)^2 + \left(\omega t - \frac{1}{4}\sin\omega t\right)^2\right)$$

c) Transitions to state $|4\rangle$ is only possible in the second order of the perturbation theory

$$c_n^{(2)}(t) = \left(\frac{1}{i\hbar}\right)^2 \sum_k \int_0^t dt' \int_0^{t'} H'_{nk}(t')\, H'_{k0}(t'')\, e^{i(n-k)\omega t'}\, e^{ik\omega t''}\, dt'' =$$

$$= -\frac{1}{\hbar^2}\frac{m^2\omega^4\gamma^2}{4}\sum_k \langle n|x^2|k\rangle\langle k|x^2|0\rangle \int_0^t \cos 2\omega t'\, e^{i(n-k)\omega t'}\left[\int_0^{t'}\cos 2\omega t''\, e^{ik\omega t''}\, dt''\right] dt' =$$

$$= -\frac{m^2\omega^4\gamma^2}{4\hbar^2}\frac{\hbar^2}{4m^2\omega^2}\sum_k \left[\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\delta_{nk}\right]\left[\sqrt{2}\,\delta_{k2} + \delta_{k0}\right] \times$$

$$\times \int_0^{t'} \cos 2\omega t'\, e^{i(n-k)\omega t'}\left[\int_0^{t'}\cos 2\omega t''\, e^{ik\omega t''}\, dt''\right] dt'$$

For $n=4$ we can simplify this to:

$$c_4^{(2)}(t) = -\frac{\omega^2\gamma^2}{16}\, 2\sqrt{6}\int_0^t \cos 2\omega t'\, e^{2i\omega t'}\left[\int_0^{t'}\cos 2\omega t''\, e^{2i\omega t''}\, dt''\right] dt' =$$

$$= -\frac{\omega^2\gamma^2\sqrt{6}}{16}\underbrace{\int_0^t \cos 2\omega t'\, e^{2i\omega t'}\left[\frac{e^{4i\omega t'}-1}{4i\omega} + t\right] dt'}_{-\frac{1}{64\omega^2}\left(-1+e^{4i\omega t}+4i\omega t\right)^2}$$

$$C_4^{(2)}(t) = \frac{\gamma^2\sqrt{6}}{1024}\left(-1+e^{4i\omega t}+4i\omega t\right)^2 = \frac{\gamma^2\sqrt{6}}{1024}\left[-1+\cos 4\omega t+i(\sin 4\omega t+4\omega t)\right]^2$$

$$= \frac{\gamma^2\sqrt{6}}{1024}\left[(\cos 4\omega t-1)^2-(\sin 4\omega t+4\omega t)^2+2i(\cos 4\omega t-1)(\sin 4\omega t+4\omega t)\right]$$

$$P_{0\to 4}^{(2)}(t) = \left|C_4^{(2)}(t)\right|^2 = \frac{3\gamma^4}{524288}\left[(\cos 4\omega t-1)^2+(\sin 4\omega t+4\omega t)^2\right]^2$$

2. The delta-function potential,

$$V(x) = -\alpha\delta(x) \qquad \alpha > 0$$

allows for a single bound state (regardless of α value)

$$\psi_\alpha(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha}{\hbar^2}|x|} \quad \text{with the energy } E = -\frac{m\alpha^2}{2\hbar^2}$$

(see Griffiths chapter 2.5.2)

The probability that the particle flies away is equal to unity minus the probability of staying in a bound state:

$$P_{leaving} = 1 - P_{bound}$$

Now

$$P_{bound} = |\langle\psi_\alpha|\psi_\beta\rangle|^2 = \frac{m^2\alpha\beta}{\hbar^4}\left[\int_{-\infty}^{+\infty} e^{-\frac{m\alpha}{\hbar^2}|x|} e^{-\frac{m\beta}{\hbar^2}|x|} dx\right]^2 =$$

$$= 4\frac{m^2\alpha\beta}{\hbar^4}\left[\int_0^{\infty} e^{-\frac{m(\alpha+\beta)}{\hbar^2}x} dx\right]^2 = 4\frac{m^2\alpha\beta}{\hbar^4}\frac{\hbar^4}{m^2(\alpha+\beta)^2} = \frac{4\alpha\beta}{(\alpha+\beta)^2}$$

$$P_{leaving} = 1 - \frac{4\alpha\beta}{(\alpha+\beta)^2} = \frac{(\alpha-\beta)^2}{(\alpha+\beta)^2}$$

3. a) If the potential, $V(r)$, is weak then the incident wave is not distorted significantly and we can use the Born approximation:

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r V(r) \sin(qr)\, dr \qquad \text{where} \quad q = 2k \sin\frac{\theta}{2}$$

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r\alpha\delta(r-R)\sin(qr)\, dr = -\frac{2m\alpha R}{\hbar^2 q}\sin qR$$

b) Low-energy limit: $k = \frac{\sqrt{2mE}}{\hbar} \to 0 \Rightarrow qR \ll 1$

$$f(\theta) \approx -\frac{2m\alpha}{\hbar^2 q} R \cdot qR = -\frac{2m\alpha R^2}{\hbar^2}$$ $\leftarrow$ note no dependence on θ

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4m^2\alpha^2 R^4}{\hbar^4}$$

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{16\pi m^2 \alpha^2 R^4}{\hbar^4}$$

4. In the Born approximation the scattering amplitude is given by

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\vec{r}\,'} V(\vec{r}\,')\,d\vec{r}\,' \qquad \vec{q} = 2k\sin\frac{\theta}{2}$$

Scattering amplitude from a single U_1 is

$$f_1 = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}\,'} U_1(\vec{r}\,')\,d\vec{r}\,'$$

Scattering amplitude from two centers $U_2 = U_1(\vec{r}) + U_1(\vec{r}+\vec{a})$ is

$$f_2 = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}\,'}\left[U_1(\vec{r}) + U_1(\vec{r}-\vec{a})\right]d\vec{r}\,' = f_1 - \frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}\,'} U_1(\vec{r}\,'-\vec{a})\,d\vec{r}\,'$$

$$= f_1 - \frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot(\vec{r}\,''+\vec{a})} U_1(\vec{r}\,'')\,d\vec{r}\,'' = f_1 + e^{i\vec{q}\cdot\vec{a}} f_1 = f_1\left(1+e^{i\vec{q}\cdot\vec{a}}\right)$$

a) Since $q^2 = 2k^2(1-\cos\theta)$ when $ka \ll 1$ then $qa \ll 1$

The scattering amplitude is then

$$f_2 \approx f_1(1+1+\dots) \approx 2f_1$$

and

$$\frac{d\sigma_2}{d\Omega} = |f_2|^2 \approx 4|f_1|^2$$

Hence

$$\frac{d\sigma_2}{d\Omega} \approx 4\frac{d\sigma_1}{d\Omega} \implies \sigma_2 = 4\sigma_1 \text{ (total cross section)}$$

b) When $kR \sim 1$ and $a \gg R$ $\quad ka \gg 1$

The term $e^{i\vec{q}\cdot\vec{a}}$ will oscillate rapidly upon slight changes in θ (and $\vec{q}$)

$$\frac{d\sigma_2}{d\Omega} = |f_2|^2 = |f_1|^2\left(1+e^{-i\vec{q}\cdot\vec{a}}\right)\left(1+e^{i\vec{q}\cdot\vec{a}}\right) = 2|f_1|^2\left(1+\cos\vec{q}\cdot\vec{a}\right) = 2\frac{d\sigma_1}{d\Omega}(1+\cos\vec{q}\cdot\vec{a})$$

For total cross section the last term ($\cos\vec{q}\cdot\vec{a}$) will not contribute due to rapid oscillations: $\sigma_2 = 2\sigma_1$