1. $E_1 = E_2 \geq E_3$

ψ_3 has more independent terms (basis functions) and obviously should yield a lower upper bound (or at least not a higher one). ψ_1 and ψ_2 yield the same upper bound because they both need to be normalized and each of them effectively has just one independent adjustable parameter.

2. $$\psi(x) = \begin{cases} Ax, & 0 \leq x \leq \frac{a}{2} \\ A(a-x), & \frac{a}{2} \leq x \leq a \\ 0, & \text{otherwise} \end{cases}$$ See lecture #1

$$1 = |A|^2 \left[\int_0^{a/2} x^2 dx + \int_{a/2}^{a} (a-x)^2 dx \right] = |A|^2 \frac{a^3}{12} \Rightarrow A = \frac{2}{a}\sqrt{\frac{3}{a}}$$

$$\frac{d\psi}{dx} = \begin{cases} A, & 0 < x < \frac{a}{2} \\ -A, & \frac{a}{2} < x < a \\ 0, & \text{otherwise} \end{cases} \qquad \frac{d^2\psi}{dx^2} = A\delta(x) - 2A\delta\left(x - \frac{a}{2}\right) + A\delta(x-a)$$

$$\langle H \rangle = -\frac{\hbar^2 A}{2m} \int \left[\delta(x) - 2\delta\left(x-\frac{a}{2}\right) + \delta(x-a) \right] \psi(x)\, dx =$$

$$= -\frac{\hbar^2 A}{2m} \left[\psi(0) - 2\psi\left(\frac{a}{2}\right) + \psi(a) \right] = \frac{\hbar^2 A^2 a}{2m} = \frac{12\hbar^2}{2ma^2}$$

$$E_{exact} = \frac{\pi^2 \hbar^2}{2ma^2} < \frac{12\hbar^2}{2ma^2}$$