The system has one level that is two-fold degenerate and one nondegenerate level. Its Hamiltonian, H°, in the basis of its eigenstates $\psi_n^{(0)}$ looks as follows:

$$H^{\circ} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \psi_1^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \psi_2^{(0)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \psi_3^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The matrix of the perturbation Hamiltonian is

$$H' = a\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

First order energy correction for the nondegenerate level (E_3) is straightforward:

$$E_3^{(1)} = H'_{33} = a$$

To get $E_1^{(1)}$ and $E_2^{(1)}$ we need to solve a 2×2 eigenvalue problem, which will also give us the correct linear combinations for the zero-order wave functions, $\phi_1^{(0)}$ and $\phi_2^{(0)}$:

$$\begin{pmatrix} H'_{11}-\varepsilon & H'_{12} \\ H'_{21} & H'_{22}-\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad \begin{pmatrix} a-\varepsilon & a \\ a & a-\varepsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad \begin{matrix} (a-\varepsilon)^2 = a^2 \\ \varepsilon = 0, 2a \end{matrix}$$

So $E_1^{(1)} = 0$ and $E_2^{(1)} = 2a$

The corresponding eigenvectors are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The correct zero-order basis for the original problem is then

$$\phi_1^{(0)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \phi_2^{(0)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \psi_3^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The second order correction to the energy is

$$E_n^{(2)} = \sum_{m\neq n} \frac{|H'_{mn}|^2}{E_n^{(0)} - E_m^{(0)}} \longleftarrow$$ here H'_{mn} are the matrix elements of H' in the correct basis.

$$E_1^{(2)} = \frac{|H'_{31}|^2}{1-2} \qquad H'_{31} = (0\ 0\ 1)\, a\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 \quad \text{so } E_1^{(2)} = 0$$

$$E_2^{(2)} = \frac{|H'_{32}|^2}{1-2} \qquad H'_{32} = (0\ 0\ 1)\, a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2}\, a$$

so $E_2^{(2)} = -2a^2$

Lastly,

$$E_3^{(2)} = \frac{|H'_{13}|^2}{2-1} + \frac{|H'_{23}|^2}{2-1} = 0 + 2a^2 = 2a^2$$

Hence to the second order the energies are:

$$E_1 = 1 + O(a^3)$$

$$E_2 = 1 + 2a - 2a^2 + O(a^3)$$

$$E_3 = 2 + a + 2a^2 + O(a^3)$$