1) In the given two-state basis our Hamiltonian is

$$H = \begin{pmatrix} \langle 1|H|1\rangle & \langle 1|H|2\rangle \\ \langle 2|H|1\rangle & \langle 2|H|2\rangle \end{pmatrix} = A\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} B(t) & A \\ A & -B(t) \end{pmatrix}$$

while $|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

In order for $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to be the eigenstates of H a $t=\pm\infty$ the latter has to be diagonal (i.e. approach a diagonal form). In other words, $|B(t=\pm\infty)| >> A$, so that we can essentially neglect A at $t=\pm\infty$

Suppose that at $t=-\infty$ the system is in state $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. It may be either the lowest or the highest state in energy (it depends on the sign of $B(-\infty)$). According to the adiabatic approximation, upon evolving from $t=-\infty$ to $t=+\infty$ the n-th state remains the n-th state. That is the lowest state remains the lowest, the highest state remains the highest. Then

$$\begin{pmatrix} B(-\infty) & 0 \\ 0 & -B(-\infty) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = B(-\infty)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

sign of $B(-\infty)$ and $B(+\infty)$ must be the same for the state to remain the lowest or the highest

$$\begin{pmatrix} B(+\infty) & 0 \\ 0 & -B(+\infty) \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -B(+\infty)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence, $B(-\infty)$ must be of opposite sign as $B(+\infty)$. The simplest function that grows to infinity by magnitude at $t=\pm\infty$ (so that we neglect A) and changes sign is a linear one: $B(t) = \alpha t$ ($\alpha = \text{const}$)

However, other choices are possible as well:

$$B(t) = \beta t^3 \qquad \beta = \text{const}$$

$$B(t) = \gamma \sinh(\lambda t) \qquad \gamma, \lambda = \text{const}$$

$$B(t) = \mu \tanh(\xi t) \qquad \mu, \xi = \text{const} \qquad \mu \gg A$$

etc.

2) The transformation of $H(t)$ as a function of time must be slow for the adiabatic approximation to hold. If we assume, for certainty, that $B(t) = \alpha t$ then

$$H = \begin{pmatrix} \alpha t & A \\ A & -\alpha t \end{pmatrix}$$

The eigenvalues of this Hamiltonian are $E_{1,2} = \pm\sqrt{A^2 + \alpha^2 t^2}$

The gap between the eigenvalues is smallest at $t = 0$ and is equal to $\Delta E = 2|A|$

The adiabatic theorem says the transitions to other states vanish in the limit $T_e \gg T_i$, where T_e is the characteristic time for changes in the Hamiltonian (in our case $T_e \approx \left|\frac{A}{\alpha}\right|$) and T_i is the characteristic time for changes in the wave function (in our case $T_i \approx \frac{\hbar}{E_2 - E_1} = \frac{\hbar}{\Delta E}$). Thus, we have

$$\left|\frac{A}{\alpha}\right| \gg \frac{\hbar}{|A|} \qquad \text{or} \qquad \alpha \ll \frac{|A|^2}{\hbar}$$