

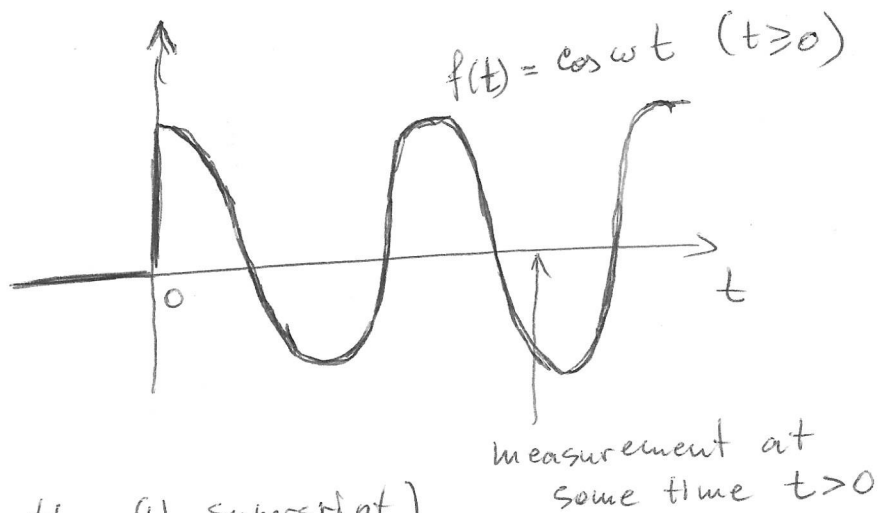
Harmonic perturbation

As an application of time-dependent perturbation theory let us consider a perturbation that is turned on at $t=0$ and is subsequently monochromatically harmonic in time. The perturbation acts on a system whose Hamiltonian is H^0 . Such a model is relevant to describe an atom that interacts with a (weak) electromagnetic field. The explicit form of the perturbation is

$$H'(\vec{r}, t) = \begin{cases} 0, & t < 0 \\ \mathcal{H}'(\vec{r}) \cos \omega t, & t \geq 0 \end{cases}$$

Using the result of the previous lecture, namely

$$c_{fi}^{(1)}(t) = \frac{\mathcal{H}'_{fi}}{i\hbar} \int_{t_0}^t e^{i\omega_{fi}t'} f(t') dt'$$



we get (we will drop the (1) superscript)

$$c_{fi}(t) = \frac{\mathcal{H}'_{fi}}{i\hbar} \int_0^t e^{i\omega_{fi}t'} \left(\frac{e^{-i\omega t'} + e^{i\omega t'}}{2} \right) dt' =$$

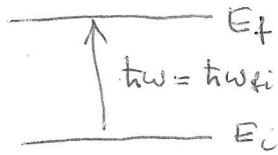
$$= -\frac{\mathcal{H}'_{fi}}{2\hbar} \left[\frac{e^{i(\omega_{fi}-\omega)t} - 1}{\omega_{fi}-\omega} + \frac{e^{i(\omega_{fi}+\omega)t} - 1}{\omega_{fi}+\omega} \right]$$

If we use the relation $e^{ix} - 1 = 2ie^{i\frac{x}{2}} \sin \frac{x}{2}$ we can rewrite it as

$$c_{fi}(t) = -\frac{i\mathcal{H}'_{fi}}{\hbar} \left[\frac{e^{i(\omega_{fi}-\omega)t/2} \sin \frac{(\omega_{fi}-\omega)t}{2}}{\omega_{fi}-\omega} + \frac{e^{i(\omega_{fi}+\omega)t/2} \sin \frac{(\omega_{fi}+\omega)t}{2}}{\omega_{fi}+\omega} \right]$$

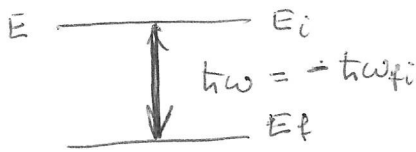
c_{fi} is big when $\omega \approx \pm \omega_{fi}$ (resonant frequency)

In the case $\omega \approx +\omega_{fi}$ $E_f > E_i$ ← system absorbs energy and "jumps" to a higher energy level $E_f = E_i + \hbar\omega$



Resonant absorption

In the case $\omega \approx -\omega_{fi}$ $E_f < E_i$ ← perturbation induces a decay in energy $E_f = E_i - \hbar\omega$

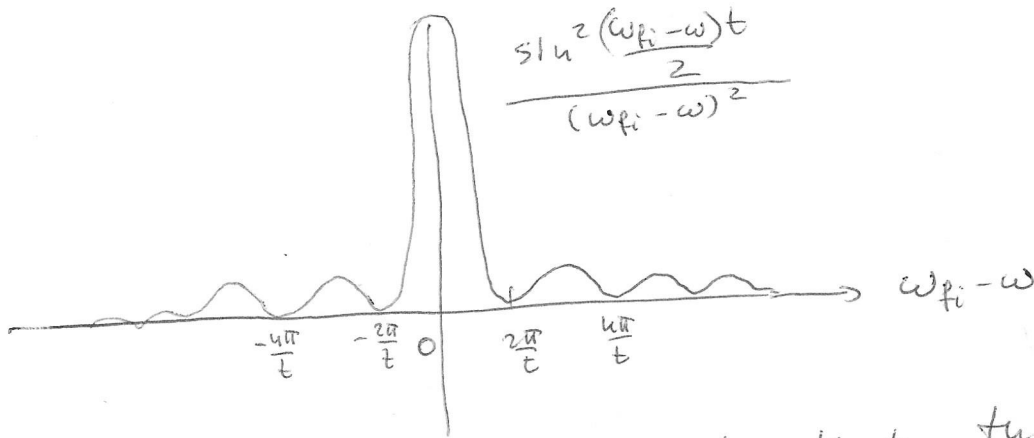


Stimulated emission

The decay process is stimulated by a photon of the same frequency in the perturbation field (stimulated emission)

Let us consider the case $\omega_{fi} > 0$. Under these conditions the first term dominates and the probability of transition is

$$P_{fi} = |C_{fi}|^2 = \frac{|\mathcal{H}'_{fi}|^2}{\hbar^2 (\omega_{fi} - \omega)^2} \sin^2 \left[\frac{(\omega_{fi} - \omega)t}{2} \right]$$



From this sketch it is evident that the states falling in the interval $|\hbar\omega_{fi} - \hbar\omega| = |E_f - (E_i + \hbar\omega)| \leq \frac{2\pi\hbar}{t} \approx \Delta E$ have the greatest probability of being excited, after the perturbation has acted for t seconds. Hence, the above inequality provides the uncertainty of the energies (after time t) that will be observed:

$$\Delta E \approx \frac{\hbar}{t} \quad E_f \approx E_i + \hbar\omega \pm \Delta E$$

Our analysis has returned the conservation of energy modified by the uncertainty relation

Let us now consider long-time evolution. The expression for P_{fi} in the limit $t \rightarrow \infty$ (or, alternatively, $\omega \rightarrow \infty$) becomes

$$P_{fi} \rightarrow \frac{\pi t}{2\hbar^2} |\mathcal{H}'_{fi}|^2 \delta(\omega_{fi} - \omega)$$

because

$$\delta(\omega) = \frac{2}{\pi} \lim_{t \rightarrow \infty} \frac{\sin^2 \frac{\omega t}{2}}{t \omega^2}$$

The corresponding transition probability rate is

$$W_{fi} = \frac{\pi}{2\hbar^2} |\mathcal{H}'_{fi}|^2 \delta(\omega_{fi} - \omega)$$

In the above formulae for W_{fi} and P_{fi} the presence of the δ -function constitutes the fact that in the long-time limit the Fourier transform of the perturbation becomes sharply peaked around the frequency of perturbation. Thus, the system will only see a single frequency. Since the uncertainty in energy $\frac{\hbar}{t}$ vanishes in this limit, the argument of the delta-function is also an expression for the conservation of energy.

In the short-time approximation, $(\omega_{fi} - \omega)t \ll 1$, the expression for P_{fi} may be expanded into a Taylor series in t :

$$P_{fi} = \frac{t^2 |\mathcal{H}'_{fi}|^2}{4\hbar^2} \quad \text{and then} \quad W_{fi} = \frac{t |\mathcal{H}'_{fi}|^2}{4\hbar^2}$$

At early times the rate at which transitions to the f -th state occur grows linearly with time.