# The Born Approximation

The SE, $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$, can be formally written as $(\nabla^2+k^2)\psi = Q$ – the Helmholz equation where $k = \frac{\sqrt{2mE}}{\hbar}$ $Q = \frac{2m}{\hbar^2}V\psi$. The inhomogeneity here depends on $\psi$, however.

Suppose now that we could solve such an equation when the inhomogeneity is a delta-function:

$$(\nabla^2+k^2)\,G(r) = \delta(\vec{r}) \equiv \delta(x)\delta(y)\delta(z)$$

$G(r)$ is called the Green's function. The solution of $(\nabla^2+k^2)\psi = Q$ could be then expressed as

$$\psi(\vec{r}) = \int G(\vec{r}-\vec{r}_0)\,Q(\vec{r}_0)\,d\vec{r}_0$$

Indeed, the direct substitution yields

$$(\nabla^2+k^2)\psi(\vec{r}) = \int\left[(\nabla^2+k^2)\,G(\vec{r}-\vec{r}_0)\right]Q(\vec{r}_0)\,d\vec{r}_0 =$$

$$= \int \delta(\vec{r}-\vec{r}_0)\,Q(\vec{r}_0)\,d\vec{r}_0 = Q(\vec{r})$$

Let us then solve the equation for $G(\vec{r})$. One way of doing it is by taking the Fourier transform

$$G(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int e^{i\vec{s}\cdot\vec{r}}\,g(\vec{s})\,d\vec{s}$$

$$(\nabla^2+k^2)G(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int\left[(\nabla^2+k^2)\,e^{i\vec{s}\cdot\vec{r}}\right]g(\vec{s})\,d\vec{s} =$$

$$= \frac{1}{(2\pi)^{3/2}}\int(-s^2+k^2)\,e^{i\vec{s}\cdot\vec{r}}\,g(\vec{s})\,d\vec{s} = \delta(\vec{r}) = \frac{1}{(2\pi)^3}\int e^{i\vec{s}\vec{r}}\,d\vec{s}$$

Clearly

$$g(\vec{s}) = \frac{1}{(2\pi)^{3/2}}\,\frac{1}{k^2-s^2}$$

Now we do the inverse Fourier transform

$$G(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\vec{s}) e^{i\vec{s}\vec{r}} d\vec{s} = \frac{1}{(2\pi)^3} \int \frac{e^{i\vec{s}\cdot\vec{r}}}{k^2 - s^2} d\vec{s} = \frac{1}{(2\pi)^3} \cdot 2\pi \cdot \int_0^{\infty}\int_0^{\pi} \frac{e^{isr\cos\theta}}{k^2 - s^2} s^2 \sin\theta \, d\theta \, ds$$

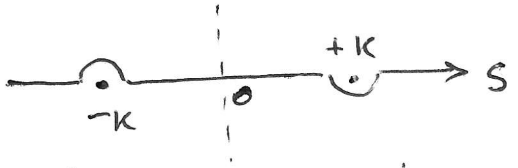
The integral over $\theta$ is : $\int_0^{\pi} e^{isr\cos\theta} \sin\theta \, d\theta = \frac{2\sin sr}{sr}$

Then

$$G(\vec{r}) = \frac{1}{(2\pi)^2} \frac{2}{r} \int_0^{\infty} \frac{s \sin sr}{k^2 - s^2} ds = \frac{1}{4\pi^2 r} \underbrace{\int_{-\infty}^{+\infty} \frac{s \sin sr}{k^2 - s^2} ds}_{I} = \frac{1}{4\pi^2 r} I = \frac{1}{4\pi r} e^{ikr}$$

The above integral $I$ can be evaluated as follows

$$I = \int_{-\infty}^{+\infty} \frac{s \sin sr}{k^2 - s^2} ds = -\int_{-\infty}^{+\infty} \frac{s \cdot \frac{e^{isr} - e^{-isr}}{2i}}{(s-k)(s+k)} ds$$




We will use a certain way to skirt poles as shown above.
It turns out this way will give the so-called retarded Green's function that propagates wave function forward in time.

$$I = -\frac{1}{2i} \int_{-\infty}^{+\infty} \frac{s e^{isr}}{(s-k)(s+k)} ds + \frac{1}{2i} \int_{-\infty}^{+\infty} \frac{s e^{-isr}}{(s-k)(s+k)} ds$$

To evaluate these two integrals we can integrate over closed countours and use the residue theorem, which says that an integral over closed contour $\gamma$ (counterclockwise) is equal to the sum of residues within the countour :

$$\int_{\gamma} f(z) dz = 2\pi i \sum_a \text{Res}[f, a]$$

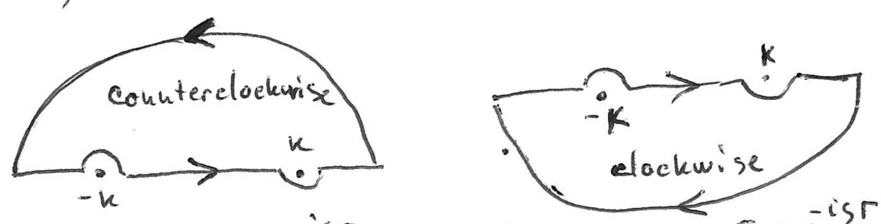
where

$$\text{Res}[f, a] = \lim_{z \to a} (z-a) f(z) \quad \text{- for a simple pole (order 1)}$$

$$\text{Res}[f, a] = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z) \quad \text{- for a pole of order } n$$

We chose the countours for the above two integrals in such a way that the semicircle at infinity contributes nothing:

With this we have:




$$I = 2\pi i \left( -\frac{1}{2i} \text{Res}\left[\frac{s e^{isr}}{(s+k)(s-k)}, +k\right] + \frac{1}{2i} \text{Res}\left[\frac{s e^{-isr}}{(s+k)(s-k)}, -k\right] \right) = -\pi \left[ \frac{k}{2k} e^{ikr} + \frac{k}{2k} e^{ikr} \right] = -\pi e^{ikr}$$

Note that we can add any $G_0(\vec{r})$ to $G(\vec{r})$ that satisfies the homogeneous Helmholz equation:

$$(\nabla^2+k^2)G_0(\vec{r})=0$$

The general solution of the Schrödinger equation then can be written as

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0)\psi(\vec{r}_0)\,d\vec{r}_0$$

where $\psi_0$ satisfies the free-particle SE:

$$(\nabla^2+k^2)\psi_0=0$$

Note that the above equation is not really a solution but an integral form of the Schrödinger equation. It is called the Lippmann-Schwinger equation. In more advanced courses it is often written as

$$|\psi^\pm\rangle = |\phi\rangle + \frac{1}{E-H_0\pm i\epsilon}V|\psi^\pm\rangle$$

The advantage of the Lippmann-Schwinger equation for scattering problems comes from two things i) it incorporates the boundary conditions ii) it can be solved iteratively.

In the first Born approximation we just plug the free-particle solution into the integral and get $\psi(\vec{r})$. The physical motivation is as follows. Suppose $V(\vec{r}_0)$ is localized around $\vec{r}_0=0$. Then when $|\vec{r}| \gg |\vec{r}_0|$ we can write

$$|\vec r - \vec r_0|^2 = r^2 + r_0^2 - 2\vec r\cdot\vec r_0 \approx r^2\left(1 - 2\frac{\vec r\cdot\vec r_0}{r^2}\right)$$

$$|r - r_0| \approx r - \frac{\vec r}{|\vec r|}\cdot\vec r_0 = r - \hat n\cdot\vec r_0$$

Let $\vec k \equiv k\hat r$ then $e^{ik|\vec r - \vec r_0|} \simeq e^{ikr}e^{-i\vec k\cdot\vec r_0}$

and $$\frac{e^{ik|\vec r - \vec r_0|}}{|\vec r - \vec r_0|} \approx \frac{e^{ikr}}{r}e^{-i\vec k\cdot\vec r_0}$$

For scattering $\psi_0(\vec r) = Ae^{ikz}$. In the case of large $r$ then

$$\psi(\vec r) \approx Ae^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec k\vec r_0}V(\vec r_0)\psi(\vec r_0)\,d\vec r_0$$

From here it is easy to deduce what $f(\theta,\varphi)$ is (scattering amplitude):

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2 A}\int e^{-i\vec k\vec r_0}V(\vec r_0)\psi(\vec r_0)\,d\vec r_0$$

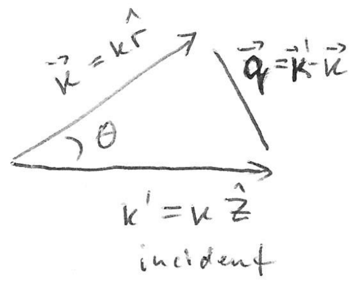
The first Born approximation makes an assumption that the incoming plane wave is not altered substantionally:

$$\psi(\vec r_0) \approx \psi_0(\vec r_0) = Ae^{ikz_0} = Ae^{i\vec k'\cdot\vec r_0}$$ where $\vec k' \equiv k\hat z$

Then

$$f(\theta,\varphi) \approx -\frac{m}{2\pi\hbar^2}\int e^{i(\vec k' - \vec k)\cdot\vec r_0}V(\vec r_0)\,d\vec r_0$$

$$= -\frac{m}{2\pi\hbar^2}\int e^{i\vec q\cdot\vec r_0}V(\vec r_0)\,d\vec r_0$$




In particular for low energy scattering ($e^{i\vec q\cdot\vec r_0} \approx 1$)

$$f(\theta,\varphi) \approx -\frac{m}{2\pi\hbar^2}\int V(\vec r_0)\,d\vec r_0$$

For a spherically symmetric potential $V(\vec r) = V(r)$

$\vec q = \vec k' - \vec k$ assume $(\vec k' - \vec k)\cdot\vec r_0 = q r_0\cos\theta_0$

Then $$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{iqr_0\cos\theta_0}V(r_0)\,r_0^2\sin\theta_0\,dr_0\,d\theta_0\,d\phi_0 =$$

$$= -\frac{2m}{\hbar^2 q}\int_0^\infty r\,V(r)\sin(qr)\,dr \qquad \text{where} \quad q = 2k\sin\frac{\theta}{2}$$

Example : Yukawa potential

$$V(r) = \beta\frac{e^{-\mu r}}{r}$$ . In the first Born approximation

$$f(\theta) \approx -\frac{2m\beta}{\hbar^2 q}\int_0^\infty e^{-\mu r}\sin(qr)\,dr = -\frac{2m\beta}{\hbar^2(\mu^2+q^2)}$$

here we used :

$$\mathrm{Im}\left[\int_0^\infty e^{-\mu r}\sin qr\,dr\right] = \mathrm{Im}\left[\int_0^\infty e^{-\mu r}e^{iqr}\,dr\right] = \mathrm{Im}\left[\frac{1}{\mu-iq}\right] = \frac{q}{\mu^2+q^2}$$