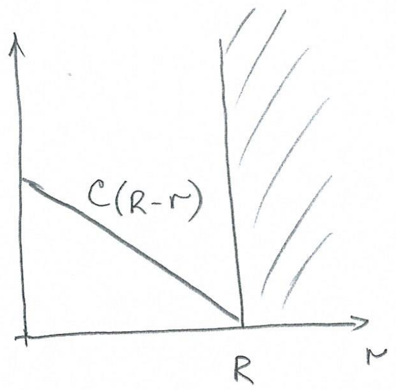
(1) Since the wave function must vanish at the well boundary $(r = R)$, our trial wave function can be written as



$$\psi(r) = \begin{cases} C(R-r), & r<b \\ 0, & r>b \end{cases}$$

This trial wave function does not have any free adjustable parameters because the value of the constant C comes from the normalization condition:

$$1 = \int |\psi|^2 d\vec{r} = |C|^2 4\pi \int_0^R (R-r)^2 r^2 dr = |C|^2 4\pi \int_0^R (r^4 - 2Rr^3 + R^2 r^2) dr =$$

$$= |C|^2 4\pi \left(\frac{R^5}{5} - \frac{1}{2}R^5 + \frac{1}{3}R^5 \right) = |C|^2 \frac{2\pi}{15} R^2 \quad \text{or} \quad |C|^2 = \frac{15}{2\pi R^5}$$

The expectation value of the Hamiltonian is:

$$E = \langle H \rangle = \int \psi \left(-\frac{\hbar^2}{2m} \nabla^2 \right) \psi \, d\vec{r} = -|C|^2 \frac{\hbar^2}{2m} 4\pi \int_0^R (R-r) \left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} (R-r) \right] r^2 dr$$

$$= \frac{15}{2\pi R^5} \frac{\hbar^2 \cdot 4\pi}{2m} \int_0^R (R-r) \left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \right] r^2 dr = \frac{15\hbar^2}{mR^5} 2 \int_0^R (R-r) r \, dr =$$

$$= \frac{30\hbar^2}{mR^5} \left(\frac{R^3}{2} - \frac{R^3}{3} \right) = \frac{5\hbar^2}{mR^2}$$

(2) The ground state is non-degenerate with both quantum numbers equal to unity, i.e. $n_x = n_y = 1$. The zero-order wave function is

$$\psi^{(0)} = \frac{2}{a} \sin\frac{\pi x}{a} \sin\frac{\pi y}{a}$$

The first-order correction to the energy is

$$E^{(1)} = \langle \psi^{(0)} | V | \psi^{(0)} \rangle = \frac{4}{a^2}\beta \int_{a/4}^{3a/4} \left(\sin\frac{\pi x}{a}\right)^2 dx \int_{a/4}^{3a/4} \left(\sin\frac{\pi y}{a}\right)^2 dy$$

Now $\displaystyle\int_{a/4}^{3a/4} \left(\sin\frac{\pi x}{a}\right)^2 dx = \int_{a/4}^{3a/4} \frac{1-\cos\frac{2\pi x}{a}}{2} dx = \frac{1}{2}\left(\frac{3a}{4} - \frac{a}{4}\right) - \frac{1}{2}\frac{a}{2\pi}\left(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2}\right) = \frac{a}{4}\left(1+\frac{2}{\pi}\right)$

Hence, $\displaystyle E^{(1)} = \frac{4}{a^2}\beta \, \frac{a}{4}\left(1+\frac{2}{\pi}\right) \frac{a}{4}\left(1+\frac{2}{\pi}\right) = \frac{\beta}{4}\left(1+\frac{2}{\pi}\right)^2$

The first excited energy level is doubly degenerate:

$n_x = 1 \quad n_y = 2 \qquad \psi_1 = \frac{2}{a} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{a}$

$n_x = 2 \quad n_y = 1 \qquad \psi_2 = \frac{2}{a} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{a}$

The matrix elements of the perturbation is this basis are

$$V_{11} = \frac{4}{a^2}\beta \underbrace{\int_{a/4}^{3a/4} \left(\sin\frac{\pi x}{a}\right)^2 dx}_{\frac{a}{4}\left(1+\frac{2}{\pi}\right)} \underbrace{\int_{a/4}^{3a/4} \left(\sin\frac{2\pi y}{a}\right)^2 dy}_{\frac{a}{4}} = \frac{\beta}{4}\left(1+\frac{2}{\pi}\right)$$

$V_{22} = V_{11} = \frac{\beta}{4}\left(1+\frac{2}{\pi}\right) \quad \leftarrow$ same as V_{11} because of symmetry $x \rightleftarrows y$

$$V_{12} = \frac{4}{a^2}\beta \underbrace{\int_{a/4}^{3a/4} \sin\frac{\pi x}{a} \sin\frac{2\pi x}{a} dx}_{0} \underbrace{\int_{a/4}^{3a/4} \sin\frac{2\pi y}{a} \sin\frac{\pi y}{a} dy}_{0} = 0$$

As we can see V is already diagonal so there is no need to diagonalize it. We can write the first-order energy corrections:

$E_1^{(1)} = V_{11} = \frac{\beta}{4}\left(1+\frac{2}{\pi}\right) \qquad E_2^{(1)} = V_{22} = \frac{\beta}{4}\left(1+\frac{2}{\pi}\right) \leftarrow$ the degeneracy is not lifted

The proper zero-order basis is the same as $\psi_{1,2}$, i.e.

$\phi_1^{(0)} = \psi_1 \qquad \phi_2^{(0)} = \psi_2$

③ a) $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$ and so $\hat{H} = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2}$

The Schrödinger equation $(\hat{H} - E)\psi = 0$ is

$$-\frac{\hbar^2}{2I}\frac{\partial^2\psi}{\partial\phi^2} - E\psi = 0 \quad \text{or} \quad \psi'' + \alpha^2\psi = 0 \quad \alpha^2 = \frac{2IE}{\hbar^2}$$

The solutions are $\psi(\phi) = A e^{\pm i\alpha\phi}$

The periodicity of ψ yields the quantization condition

$$\psi(\phi + 2\pi) = \psi(\phi) \implies 2\pi\alpha = 2\pi n \quad ; \quad n = 0, \pm 1, \pm 2, \ldots$$

Constant A is found from the normalization condition:

$$\int_0^{2\pi} \psi^*\psi\, d\phi = 1 \implies |A|^2 \int_0^{2\pi} d\phi = 1 \implies A = \frac{1}{\sqrt{2\pi}}$$

Hence the eigenfunctions of the Hamiltonian are

$$\psi_n^{(0)} = \frac{1}{\sqrt{2\pi}} e^{in\phi} \quad \text{with energy } E_n = \frac{\hbar^2 n^2}{2I} \quad n = 0, \pm 1, \pm 2, \ldots$$

Note that all energy levels except the ground one $(n=0)$ are doubly degenerate. Physically this corresponds to clockwise and counter-clockwise rotation.

b) The first-order correction to the ground energy level is

$$E_0^{(1)} = \langle \psi_0^{(0)} | H' | \psi_0^{(0)} \rangle = -\frac{\lambda}{2\pi}\int_0^{2\pi} \cos 2\phi\, d\phi = 0$$

Let us compute the second-order correction then:

$$E_0^{(2)} = \sum_{m \neq 0} \frac{|H'_{m0}|^2}{E_0^{(0)} - E_m^{(0)}} \quad \text{where } H'_{m0} = \langle \psi_m^{(0)} | H' | \psi_0^{(0)} \rangle = -\frac{\lambda}{2\pi}\int_0^{2\pi} e^{-im\phi}\cos 2\phi\, d\phi$$

The last integral is given in the appendix. With that

$$H'_{m0} = -\frac{\lambda}{2\pi}\pi(\delta_{2,m} + \delta_{2,-m}) = -\frac{\lambda}{2}(\delta_{2,m} + \delta_{2,-m})$$

Then we can see that only two terms survive in the sum

$$E_0^{(2)} = \frac{\frac{\lambda^2}{4}}{0 - \frac{4\hbar^2}{2I}} + \frac{\frac{\lambda^2}{4}}{0 - \frac{4\hbar^2}{2I}} = -\frac{\lambda^2 I}{4\hbar^2}$$

c) $$\psi_0^{(1)} = \sum_{m \neq 0} \frac{H'_{mn}}{E_0^{(0)} - E_m^{(0)}} \psi_m^{(0)}$$ Again, only two terms survive

$$\psi_0^{(1)} = \frac{-\frac{\lambda}{2}}{0-\frac{4\hbar^2}{2I}}\psi_2^{(0)} + \frac{-\frac{\lambda}{2}}{0-\frac{4\hbar^2}{2I}}\psi_{-2}^{(0)} = \frac{\lambda I}{4\hbar^2}\left(\psi_2^{(0)} + \psi_{-2}^{(0)}\right)$$

d) The first excited energy level is doubly degenerate.

Matrix elements of the perturbation are

$$\langle\psi_1^{(0)}|H'|\psi_1^{(0)}\rangle = -\frac{\lambda}{2\pi}\int_0^{\pi}\cos 2\phi\, d\phi = 0$$

$$\langle\psi_{-1}^{(0)}|H'|\psi_{-1}^{(0)}\rangle = -\frac{\lambda}{2\pi}\int_0^{\pi}\cos 2\phi\, d\phi = 0$$

$$\langle\psi_1^{(0)}|H'|\psi_{-1}^{(0)}\rangle = -\frac{\lambda}{2\pi}\int_0^{\pi} e^{-2i\phi}\cos 2\phi\, d\phi = -\frac{\lambda}{2\pi}\pi\left(\delta_{-2,2}+\delta_{2,2}\right) = -\frac{\lambda}{2}$$

So

$$H' = -\frac{\lambda}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solving the secular equation $\det(H' - E^{(1)}I) = 0$

yields:

$$E_{1,-1}^{(1)} = \pm\frac{\lambda}{2}$$

(4) Bohr-Sommerfeld quantization rule for a potential with no vertical walls:

$$\int_a^b p(x)\,dx = (n-\tfrac{1}{2})\pi\hbar \qquad n = 1,2,3,\ldots \qquad p(x) = \sqrt{2m(E-V(x))}$$

Classical turning points are found by solving $\alpha|x| = E$

$$a = -\frac{E}{\alpha} \qquad b = +\frac{E}{\alpha}$$

With that we have

$$\int_{-E/\alpha}^{+E/\alpha} \sqrt{2m(E-\alpha|x|)}\,dx = (n-\tfrac{1}{2})\pi\hbar$$

$$\int_{-E/\alpha}^{0} \sqrt{\frac{E}{\alpha}+x}\,dx + \int_0^{E/\alpha} \sqrt{\frac{E}{\alpha}-x}\,dx = \frac{1}{\sqrt{2m\alpha}}(n-\tfrac{1}{2})\pi\hbar$$

$$\frac{2}{3}\left(\frac{E}{\alpha}+x\right)^{3/2}\Bigg|_{-E/\alpha}^{0} - \frac{2}{3}\left(\frac{E}{\alpha}-x\right)^{3/2}\Bigg|_0^{E/\alpha} = \frac{1}{\sqrt{2m\alpha}}(n-\tfrac{1}{2})\pi\hbar$$

$$\frac{2}{3}\left(\frac{E}{\alpha}\right)^{3/2} + \frac{2}{3}\left(\frac{E}{\alpha}\right)^{3/2} = \frac{1}{\sqrt{2m\alpha}}(n-\tfrac{1}{2})\pi\hbar \quad \text{or} \quad \left(\frac{E}{\alpha}\right)^{3/2} = \frac{3}{4}\frac{(n-\frac{1}{2})\pi\hbar}{(2m\alpha)^{1/2}}$$

$$E_n = \frac{\left[\frac{3}{4}(n-\frac{1}{2})\pi\hbar\alpha\right]^{2/3}}{[2m]^{1/3}}$$

At large n $\quad E_n \sim n^{\frac{2}{3}}$ which makes sense because for the particle in a box $(V \sim x^{\infty})$ we have $E_n \sim n^2$ and for the harmonic oscillator $(V \sim x^2)$ we have $E_n \sim n^1$