1 The eigenstates of the unperturbed Hamiltonian are:

$$\psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \qquad n = 1,2,3,4\ldots$$

If we employ the time-dependent perturbation theory then

$$c_n^{(0)} = \delta_{n1} \quad (\text{i.e. } c_1^{(0)} = 1,\ c_2^{(0)} = 0,\ c_3^{(0)} = 0, \ldots)$$

$$c_n^{(1)}(\infty) = c_n^{(1)}(\tau) = \frac{1}{i\hbar}\int_0^\tau H'_{n1}(t')e^{i\omega_{n1}t'}dt' = \frac{1}{i\hbar}\int_0^\tau \langle\psi_n|H'|\psi_1\rangle e^{i\omega_{n1}t'}dt'$$

where $\omega_{n1} = \frac{1}{\hbar}(E_n - E_1) = \frac{\pi\hbar}{2ma^2}(n^2-1)$

Over the period from 0 to τ H' is constant in time, so

$$c_n^{(1)}(\tau) = \frac{1}{i\hbar}\int_0^\tau\left(\int_{\frac{a}{2}-\frac{b}{2}}^{\frac{a}{2}+\frac{b}{2}} \frac{2}{a}\sin\frac{n\pi x}{a}(-V_0)\sin\frac{\pi x}{a}\,dx\right)e^{i\omega_{n1}t'}dt' =$$

$$= \frac{1}{i\hbar}\,\frac{e^{i\omega_{n1}\tau}-1}{i\omega_{n1}}\left(-\frac{2V_0}{a}\right)\underbrace{\int_{\frac{a}{2}-\frac{b}{2}}^{\frac{a}{2}+\frac{b}{2}}\sin\frac{n\pi x}{a}\sin\frac{\pi x}{a}\,dx}_{I}$$

When $b \ll a$ integral I vanishes for even n values. For odd n values it is equal to: $I \approx (-1)^{\frac{n-1}{2}} b$

Hence

$$c_n^{(1)}(\tau) = (-1)^{\frac{n-1}{2}}\frac{2V_0 b}{\hbar a}\frac{(e^{i\omega_{n1}\tau}-1)}{\omega_{n1}} = (-1)^{\frac{n-1}{2}}\frac{4V_0 b\, i\, e^{\frac{i\omega_{n1}\tau}{2}}}{\hbar a\,\omega_{n1}}\sin\frac{\omega_{n1}\tau}{2} \qquad (n \text{ is odd})$$

and

$$P_2^{(1)} = |c_2^{(1)}(\tau)|^2 = 0 \qquad P_4^{(1)} = |c_2^{(1)}(\tau)|^2 = 0$$

$$P_3^{(1)} = |c_3^{(1)}(\tau)|^2 = \frac{16V_0^2 b^2}{\hbar^2 a^2 \omega_{31}^2}\sin^2\frac{\omega_{31}\tau}{2}$$

(2) The unperturbed Hamiltonian is

$$H = -\vec{\mu}\cdot\vec{B} = \frac{e}{m} B_x S_x = \mu_B B_x \sigma_x = \mu_B B_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Its eigenenergies and eigenstates are:

$$E_+ = \mu_B B_x \qquad \chi_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$E_- = -\mu_B B_x \qquad \chi_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The perturbation operator is given by

$$H' = \mu_B B_z \sigma_z = \mu_B B_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In the first order time-dependent perturbation theory we get:

$$C_-^{(1)}(\infty) = C_-^{(1)}(\tau) = \frac{1}{i\hbar}\int_0^\tau \langle \chi_- | H' | \chi_+ \rangle e^{i\omega t'} dt' \qquad \text{here } \omega = \frac{2\mu_B B_x}{\hbar}$$

$$\langle \chi_- | H' | \chi_+ \rangle = \frac{\mu_B B_z}{2} (1 \; -1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mu_B B_z$$

With that we have

$$C_-^{(1)}(\tau) = \frac{\mu_B B_z}{i\hbar}\int_0^\tau e^{i\omega t'} dt' = \frac{\mu_B B_z}{i\hbar}\,\frac{e^{i\omega\tau}-1}{i\omega} = -\frac{\mu_B B_z\, 2i}{\hbar\omega} e^{\frac{i\omega\tau}{2}} \sin\frac{\omega\tau}{2} =$$

$$= -i\frac{B_z}{B_x} e^{\frac{i\omega\tau}{2}} \sin\frac{\omega\tau}{2}$$

Then the probability of transition from χ_+ to χ_- is:

$$P_-^{(1)} = |C_-^{(1)}(\tau)|^2 = \left(\frac{B_z}{B_x}\right)^2 \sin^2\frac{\mu_B B_x \tau}{\hbar}$$

(3) a) Since we are asked to compute $\frac{d\sigma}{d\Omega}$ to the lowest order in V, it is clearly assumed that the perturbation theory is applicable. For a central potential the first Born approximation gives

$$f(\theta) = -\frac{2m}{\hbar^2 q}\int_0^\infty r V(r) \sin qr \, dr \qquad \text{where } \kappa = 2k \sin\frac{\theta}{2} \text{ and } k \text{ is}$$

related to the incident energy as follows: $k = \frac{\sqrt{2mE}}{\hbar}$

For our potential this yields

$$f(\theta) = \frac{2mV_0}{\hbar^2 q}\int_0^R r \sin qr \, dr = \frac{2mV_0}{\hbar^2 q^3}\left[\sin qr - qr\cos qr\right]\Big|_0^R =$$

$$= \frac{2mV_0}{\hbar^2 q^3}\left[\sin qR - qR\cos qR\right]$$

Then

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4m^2V_0^2}{\hbar^4 q^6}\left[\sin qR - qR\cos qR\right]^2 = \frac{4m^2V_0^2R^6}{\hbar^4}\frac{\left[\sin x - x\cos x\right]^2}{x^6}$$

$$x \equiv qR$$

b) When the incident energy is small, k is also small and so is x $(x = qR = 2Rk\sin\frac{\theta}{2})$. Then we can take a limit in the expansion of $\frac{d\sigma}{d\Omega}$ as $x \to 0$:

$$\frac{\left[\sin x - x\cos x\right]^2}{x^6} = \frac{\left[x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots - x\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right)\right]^2}{x^6} \approx \frac{1}{9} + O(x^2)$$

So

$$\frac{d\sigma}{d\Omega} \approx \frac{4}{9}\frac{m^2V_0^2R^6}{\hbar^4} \quad \text{and} \quad \sigma_{total} = \int \frac{d\sigma}{d\Omega}d\Omega = \frac{16\pi}{9}\frac{m^2V_0^2R^6}{\hbar^4}$$

This result is consistent with part (a) if $kR \ll 1$. However, at the same time the incident energy E (and k) must be large enough for the perturbation theory to be applicable (so that the potential V(r) can be treated as a perturbation)

This requires

$$\frac{\hbar^2k^2}{2m} \gg V_0 \quad \text{or} \quad k \gg \frac{\sqrt{2mV_0}}{\hbar}$$

④ The radial Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]R = ER$$

If $V(r)$ is short-range we can neglect it ar $r\to\infty$.

Then

$$r^2R'' + 2rR' + \left[k^2r^2 - \ell(\ell+1)\right]R = 0$$

This is the spherical Bessel equation. The general solution is a linear combination of either spherical Bessel/Neumann or spherical Hankel functions

$$R(r) = h_\ell^{(1)}(kr) + h_\ell^{(2)}(kr)$$

At large kr $(kr\to\infty)$ $\quad h_\ell^{(1)} \sim (-i)^{\ell+1}\frac{e^{ikr}}{kr} \qquad h_\ell^{(2)} \sim (i)^{\ell+1}\frac{e^{-ikr}}{kr}$

Since we are interested in a scattered wave $\left(\frac{e^{ikr}}{kr}\right)$ the term with $h_\ell^{(2)}(kr)$ must be dropped.

the general solution of the 3D Schrödinger equation (the scattered wave function) can be expanded as

$$\sum_{\ell m} A_{\ell m} R_\ell(kr) Y_{\ell m}(\theta,\phi) \qquad A_{\ell m} \text{ - some coefficients}$$

or, in the case of spherically symmetric potential

$$\sum_\ell B_{\ell m} h_\ell^{(1)}(kr) P_\ell(\cos\theta)$$

Since all $h_\ell^{(1)}$ go as $\sim \frac{e^{ikr}}{kr}$ as $r\to\infty$ (regardless of ℓ) it is clear that the scattered wave function has the form

$$\frac{e^{ikr}}{r} f(\theta)$$

where $f(\theta)$ is some function of θ.