Essentially we use the Rayleigh-Ritz method

$$\psi = \sum_{i=1}^{2} c_i f_i$$

The variational upper bounds to the energies, ε, and the corresponding approximate wave functions (defined by $\{c_i\}$) are found by solving the generalized eigenvalue problem

$$Hc = \varepsilon Sc$$

In our case

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \qquad S = \begin{pmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

The resulting secular equation $\det(H - \varepsilon S) = 0$ yields:

$$\begin{vmatrix} 2-\varepsilon & 1-\frac{1}{2}\varepsilon \\ 1-\frac{1}{2}\varepsilon & 4-\varepsilon \end{vmatrix} = (2-\varepsilon)(4-\varepsilon) - (1-\tfrac{1}{2}\varepsilon)^2 = \tfrac{3}{4}\varepsilon^2 + 5\varepsilon + 7 = 0$$

$$\varepsilon_{1,2} = \frac{5 \pm \sqrt{25-21}}{2 \cdot \frac{3}{4}} = 2, \frac{14}{3}$$

The smaller of the two ε's correspond to the ground state. Hence the upper bound to the ground state energy is

$\varepsilon = 2$ (in the same units as matrix elements $\langle f_i | H | f_j \rangle$ are given)