# PHYS 452 Quantum Mechanics II (Fall 2017)
# Homework #2, due Tuesday Sept 26 in class

Perturbation theory

1. A particle of mass $m$ moving in an infinite potential well of width $a$ $(0 < x < a)$ is subject to a small perturbative potential $V(x) = V_0 \cos^2 \frac{\pi x}{a}$. Find the shifts of the energy levels up to the second order in $V$.

2. A particle of mass $m$ moves in 1D potential whose form is very close to the one of a harmonic oscillator, namely
$$V(x) = \frac{m\omega^2 x^2}{2} \left(\frac{x}{a}\right)^{2\lambda}$$
where $a$ is a parameter with the dimensions of length and $\lambda \ll 1$.

   (a) Express the potential as a sum of a potential for which you know the analytic solution and a small perturbation. Calculate the ground state energy of the particle up to the first order in the perturbing potential.

   (b) Consider a trial variational wave function in the form $\psi(x) = \left(\frac{\beta}{\pi}\right)^{1/4} e^{-\beta x^2/2}$. Compute the lowest variational upper bound of the energy without assuming that $\lambda \ll 1$ and determine the optimal $\beta$ as a function of $\lambda$. *Then* make use of the smallness of $\lambda$ and expand the energy in terms of $\lambda$ up to the smallest order. Compare the result to the one you obtained in 2a. It might be useful to use the relation
$$\int_{-\infty}^{+\infty} e^{-y^2} y^{2+2\lambda} = \Gamma(\lambda + 3/2),$$
where $\Gamma(x)$ is the Euler gamma function. If you are not familiar with the gamma function, look up its definition and basic properties in a handbook or on the internet.

3. Consider a particle of mass $m$ that moves in the potential $V(x) = \frac{1}{2} m\omega^2 x^2$. Assuming that the particle moves with the velocity that is much smaller than the speed of light, $c$, allow for the relativistic correction in the relation between the kinetic energy and momentum. That is, by using the relativistic formula for the total energy, $E^2 = m^2c^4 + p^2c^2$, determine the expression for the kinetic energy up to the two leading terms in $p$ and use in the Hamiltonian. Then use the perturbation theory to find the ground state energy accurate up to order $1/c^2$.

4. An electron moves in a Coulomb potential. What is degeneracy of the the first excited energy level ($n{=}2$)? Determine what happens to this energy level when an additional weak non-central potential is turned on that has the form
$$H' = xyf(r),$$
where $f(r)$ is some arbitrary, well-behaved, spherically symmetric function that approaches zero rapidly as $r \to \infty$. How many sublevels are there in the presence of this perturbation? Is the degeneracy lifted completely? If not, what is the remaining degeneracy of each sublevel? What is the relative shift of each sublevel with respect to the nonperturbed energy?