

(1) Born-Sommerfeld quantization rule (no vertical walls):

$$\int_{r_1}^{r_2} \sqrt{2m [E - V_{\text{eff}}(r)]} dr = (n - \frac{1}{2}) \pi \hbar$$

here $V_{\text{eff}}(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} - \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2}$, n is the principal quantum number

r_1 and r_2 are classical turning points, i.e. the solutions of the equation $-\frac{Ze^2}{4\pi\epsilon_0 r} - \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} = E$, or

$$\underbrace{\frac{Ze^2}{4\pi\epsilon_0} r - \frac{\hbar^2}{2m} l(l+1)}_{\alpha} = \underbrace{Er^2}_{\beta}, \text{ or } r^2 + \frac{\alpha}{E} r + \frac{\beta}{E} = 0 \text{ or } (r-r_1)(r-r_2)=0$$

$$\text{so } r_{1,2} = -\frac{\alpha}{2E} \left(1 \pm \sqrt{1 - \frac{4\beta E}{\alpha^2}} \right)$$

Now, keeping in mind that E is negative

$$\begin{aligned} (n - \frac{1}{2}) \pi \hbar &= \sqrt{-2mE} \int_{r_1}^{r_2} \sqrt{\frac{V_{\text{eff}}}{E} - 1} dr = \sqrt{-2mE} \int_{r_1}^{r_2} \sqrt{-\frac{\alpha}{Er} - \frac{\beta}{E} - 1} dr = \\ &= \sqrt{-2mE} \underbrace{\int_{r_1}^{r_2} \frac{1}{r} \sqrt{(r-r_1)(r_2-r)} dr}_{\frac{\pi}{2} (\sqrt{r_2} - \sqrt{r_1})^2} = \sqrt{-2mE} \frac{\pi}{2} \left(-\frac{\alpha}{2E} \right) \left(\sqrt{1 - \frac{4\beta E}{\alpha^2}} - \sqrt{1 + \frac{4\beta E}{\alpha^2}} \right)^2 \\ &= \frac{\pi \alpha}{2} \sqrt{-\frac{2m}{E}} \left(1 - \sqrt{1 - \frac{4\beta E}{\alpha^2}} \right) = \frac{\pi \alpha}{2} \sqrt{-\frac{2m}{E}} \left(1 - 2\sqrt{\frac{\beta E}{\alpha^2}} \right) \end{aligned}$$

or

$$(2n-1)\hbar\sqrt{-E} = \alpha\sqrt{2m} - 2\sqrt{-2m\beta E}$$

Solving the last equation for E yields

$$\begin{aligned} \sqrt{-E} \left((2n-1)\hbar + 2\sqrt{2m\beta} \right) &= \alpha\sqrt{2m} \Rightarrow E = \frac{2m\alpha^2}{\hbar^2} \frac{1}{\left((2n-1) + \frac{2\sqrt{2m\beta}}{\hbar} \right)^2} = \\ &= \frac{m\alpha^2}{2\hbar^2} \frac{1}{\left(n - \frac{1}{2} + \frac{\sqrt{e(l+1)}}{\hbar} \right)^2} = \frac{m Z^2 e^4}{32\pi^2 \epsilon_0^2 \hbar^2} \frac{1}{\left(n - \frac{1}{2} + \frac{\sqrt{e(l+1)}}{\hbar} \right)^2} \end{aligned}$$

(2)

See lecture #19

③ The unperturbed Hamiltonian of the particle is

$$H_0 = -\vec{\mu} \cdot \vec{B} = -g \vec{B} \cdot \vec{S} = -g B_x S_x = -\frac{\hbar g B_x}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The eigenvalues and eigenstates of H_0 are:

$$E_- = \hbar g B_x \quad \Psi_- = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \quad S_x = -\hbar$$

$$E_0 = 0 \quad \Psi_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad S_x = 0$$

$$E_+ = -\hbar g B_x \quad \Psi_+ = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \quad S_x = +\hbar$$

The perturbing Hamiltonian ($0 < t < T$) is

$$H' = -g B_y S_y = -\frac{\hbar g B_y}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Initially the particle in state 2: $C_-^{(0)} = 0$ $C_0^{(0)} = 1$ $C_+^{(0)} = 0$

To determine the probability of transitions at time $t > T$ we can use the time-dependent perturbation theory formulae.

In particular,

$$C_-^{(1)}(t > T) = C_-^{(1)}(T) = \frac{1}{i\hbar} \int_0^T \langle \Psi_- | H' | \Psi_0 \rangle e^{i\omega_{-0} t'} dt' = \omega_{-0} = \frac{E_- - E_0}{\hbar} = g B_x$$

$$\langle \Psi_- | H' | \Psi_0 \rangle = -\frac{\hbar g B_y}{\sqrt{2}} \left(\frac{1}{2} - \frac{1}{\sqrt{2}} \frac{1}{2} \right) \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = -i \frac{\hbar g B_y}{\sqrt{2}}$$

so

$$C_-^{(1)}(T) = -\frac{g B_y}{\sqrt{2}} \int_0^T e^{ig B_x t'} dt' = -\frac{g B_y}{\sqrt{2}} \frac{e^{ig B_x T} - 1}{ig B_x} = \frac{i B_y}{\sqrt{2} B_x} (e^{ig B_x T} - 1) =$$

$$= \frac{i}{\sqrt{2}} \frac{B_y}{B_x} 2i e^{\frac{ig B_x T}{2}} \sin\left(\frac{g B_x T}{2}\right) = -\sqrt{2} \frac{B_y}{B_x} e^{\frac{ig B_x T}{2}} \sin\left(\frac{g B_x T}{2}\right)$$

The corresponding probability of $0 \rightarrow -$ transition is

$$P_-^{(1)}(T) = |C_-^{(1)}(T)|^2 = 2 \frac{B_y^2}{B_x^2} \sin^2\left(\frac{g B_x T}{2}\right)$$

(4) The Hamiltonian of the system is

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{k(t)x^2}{2} = \underbrace{-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{k_0 x^2}{2}}_{H_0} + \underbrace{dt e^{-\beta t} \frac{x^2}{2}}_{H'(t)}$$

In the first order of the time-dependent perturbation theory we have

$$C_n(\infty) = \frac{1}{i\hbar} \int_0^\infty \langle n | H'(t) | 0 \rangle e^{i\omega_{n0} t} dt = \frac{\alpha}{2i\hbar} \langle n | x^2 | 0 \rangle \int_0^\infty t e^{-\beta t} e^{i\omega t} dt$$

$$\text{where } \omega_{n0} = \frac{E_n - E_0}{\hbar} = \hbar\omega \quad \omega = \sqrt{\frac{k_0}{m}}$$

$$\text{Now for } n \neq 0 \text{ we set } \langle n | x^2 | 0 \rangle = \frac{\hbar^2}{2m\omega} \sqrt{2} \delta_{n,2}$$

With that the expression for C_n becomes :

$$C_n(\infty) = \frac{\alpha}{2i\hbar} \frac{t}{2m\omega} \sqrt{2} \delta_{n,2} \int_0^\infty t e^{-(\beta - i\hbar\omega)t} dt$$

$$\underbrace{\frac{1}{(\beta - i\hbar\omega)^2}}$$

$$C_2(\infty) = \frac{\alpha}{2\sqrt{2}i\hbar\omega} \frac{1}{(\beta^2 - 4\omega^2) - 4i\hbar\omega\beta}$$

$$P_2(\infty) = |C_2(\infty)|^2 = \frac{\alpha^2}{8m^2\omega^2} \frac{1}{(\beta^2 - 4\omega^2)^2 + 16\beta^2\omega^2} = \frac{\alpha^2}{8m^2\omega^2} \frac{1}{(\beta^2 + 4\omega^2)^2}$$

Probability of transition to states other than $|2\rangle$ are zeros in the first order of the perturbation theory