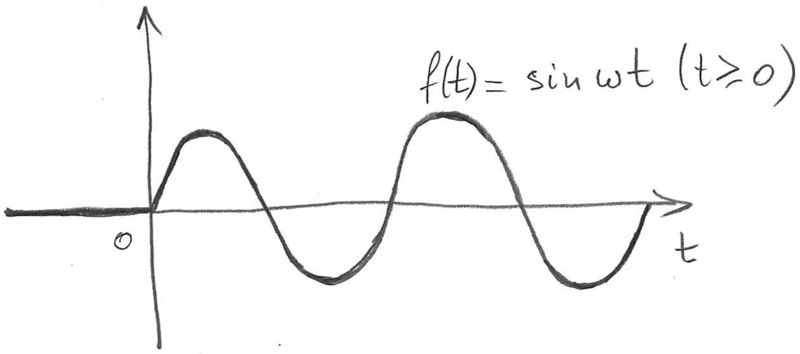
Harmonic perturbation

As an application of time-dependent perturbation theory let us consider a perturbation that is turned on at $t=0$ and is subsequently monochromatic in time. The perturbation acts on a system whose Hamiltonian is H^0. Such a model is relevant for describing an atom that interacts with a weak electromagnetic field. The explicit formula for the perturbation is



$$H'(\vec{r},t) = \begin{cases} 0, & t<0 \\ \mathcal{H}'(\vec{r})\sin\omega t, & t\geq 0 \end{cases}$$

Using the expression for the probability amplitudes obtained in the previous lecture, namely

$$c_{fi}^{(1)} = \frac{\mathcal{H}'_{fi}}{i\hbar}\int_{t_0}^{t} e^{i\omega_{fi}t'} f(t')\,dt'$$

we get (we will drop the (1) superscript for simplicity)

$$c_{fi}(t) = \frac{\mathcal{H}'_{fi}}{i\hbar}\int_0^t e^{i\omega_{fi}t'}\left(\frac{e^{i\omega t'}-e^{-i\omega t'}}{2i}\right)dt' =$$

$$= -\frac{\mathcal{H}'_{fi}}{2\hbar i}\left[\frac{e^{i(\omega_{fi}+\omega)t}-1}{\omega_{fi}+\omega} - \frac{e^{i(\omega_{fi}-\omega)t}-1}{\omega_{fi}-\omega}\right]$$

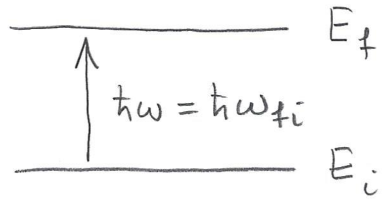
If we use the relation $e^{i\alpha}-1 = e^{\frac{i\alpha}{2}}\left(e^{\frac{i\alpha}{2}}-e^{-\frac{i\alpha}{2}}\right) = 2ie^{\frac{i\alpha}{2}}\sin\frac{\alpha}{2}$

we can rewrite it as

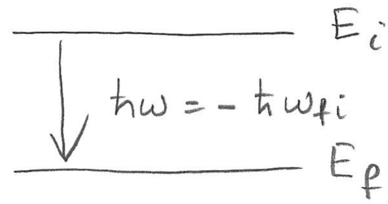
$$c_{fi}(t) = -\frac{\mathcal{H}'_{fi}}{\hbar}\left[\frac{e^{i(\omega_{fi}+\omega)t/2}\sin\frac{(\omega_{fi}+\omega)t}{2}}{\omega_{fi}+\omega} - \frac{e^{i(\omega_{fi}-\omega)t/2}\sin\frac{(\omega_{fi}-\omega)t}{2}}{\omega_{fi}-\omega}\right]$$

c_{fi} becomes large when $\omega \simeq \pm \omega_{fi}$ (resonant frequency)

In the case $\omega \simeq +\omega_{fi}$ $E_f > E_i$ and the system absorbs energy and "jumps" to a higher energy level $E_f = E_i + \hbar\omega$



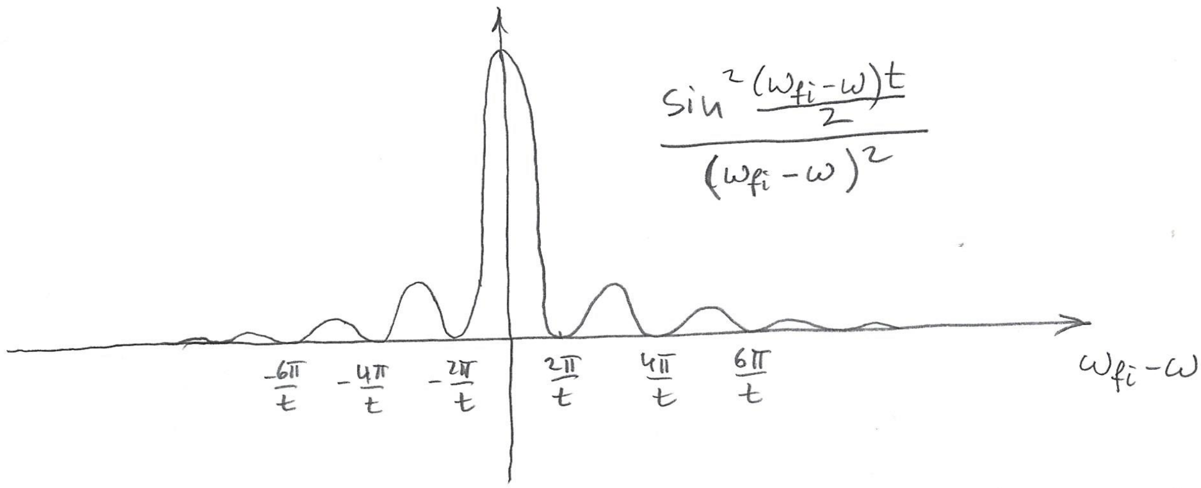
In the case $\omega \simeq -\omega_{fi}$ $E_f < E_i$ and we can say that the perturbation induces a decay in energy $E_f = E_i - \hbar\omega$



The decay process is stimulated by a photon of the same frequency in the perturbation field (stimulated emission)

Let us consider the case when $\omega_{fi} > 0$. Under this condition the second term in the expression for $c_{fi}(t)$ above dominates. The corresponding probability of transition is then

$$P_{fi} = |c_{fi}|^2 = \frac{|\mathcal{H}'_{fi}|^2}{\hbar^2(\omega_{fi}-\omega)^2} \sin^2\left[\frac{(\omega_{fi}-\omega)t}{2}\right]$$



From the sketch above it is evident that the states falling in the interval $|\hbar\omega_{fi} - \hbar\omega| = |E_f - (E_i + \hbar\omega)| \le$ $\le \frac{2\pi\hbar}{t} \approx \Delta E$ have the greatest probability of being excited, after the perturbation has acted for time t. Hence, the above inequality provides the uncertainty in the energy (after time t) that will be observed:

$$\Delta E \simeq \frac{\hbar}{t} \qquad E_f \simeq E_i + \hbar\omega \pm \Delta E$$

Our analysis has yielded the conservation of energy within the uncertainty.

Let us now consider long-time evolution. The expression for P_{fi} in the limit $t \to \infty$ (or, alternatively, $\omega \to \infty$) becomes resonant at $\omega = \pm \omega_{fi}$ and, in fact, approaches a delta function:

$$\frac{1}{\pi} \lim_{t \to \infty} \frac{\sin^2 \alpha t}{\alpha^2 t} = \delta(\alpha)$$

$$P_{fi} = \frac{|\mathcal{H}'_{fi}|^2 \frac{t}{2}}{\hbar^2 (\omega_{fi} \mp \omega) \frac{t}{2}} \sin^2 \left[\frac{(\omega_{fi} \mp \omega) t}{2} \right] \longrightarrow \frac{\pi t |\mathcal{H}'_{fi}|^2}{2\hbar^2} \delta(\omega_{fi} \mp \omega)$$

The corresponding transition probability rate is

$$W_{fi} = \frac{d P_{fi}}{dt} = \frac{\pi}{2\hbar^2} |\mathcal{H}'_{fi}| \, \delta(\omega_{fi} \mp \omega)$$

In the formulae for W_{fi} and P_{fi} the presence of the delta function constitutes the fact that in the long-time limit the Fourier transform of the perturbation becomes sharply peaked around the frequency of the perturbing field. Thus, the system will only see a single frequency. Since the uncertainty in energy, $\frac{\hbar}{t}$, vanishes in this limit, the argument of the delta function is also a manifestation of the conservation of energy.

In the short-time approximation, when $(\omega_{fi} - \omega) t \ll 1$

the expression for P_{fi} may be expanded into a Taylor series in t:

$$P_{fi} = \frac{t^2 |\mathcal{H}'_{fi}|^2}{4\hbar^2} \quad \text{and then} \quad W_{fi} = \frac{t |\mathcal{H}'_{fi}|^2}{2\hbar^2}$$

At early times the rate at which transitions to the final state occur grows linearly with time.