Quantum dynamics of a two-level atom. Rabi oscillations

The dynamics of a two-level systems is a widely encountered problem in many applications. Some systems are inherently two-level ones (e.g. the spin of a spin-1/2 particle). For some others a two-level approximation can serve as a good model that in certain cases can capture qualitative behavior very well.

Let us consider a (quantum) atom whose energy levels are $E_a = \hbar\omega_a$ and $E_b = \hbar\omega_b$ ($E_b > E_a$), so that the angular transition frequency is $\omega_{ba} = \omega_b - \omega_a$. As we saw it it one of the previous lectures the dynamics of such system is described by two coupled differential equations

$$\dot{c}_a = -\frac{i}{\hbar}\left[c_a H'_{aa} + c_b H'_{ab} e^{-i\omega_{ba}t}\right] \qquad (*)$$

$$\dot{c}_b = -\frac{i}{\hbar}\left[c_a H'_{ba} e^{i\omega_{ba}t} + c_b H'_{bb}\right]$$

where $H'_{ab} = \langle \psi_a | H'(t) | \psi_b \rangle$, etc.

(Recall that in our previous notations $H = H_0 + H'(t)$

$H_0 \psi_a = E_a \psi_a \qquad H_0 \psi_b = E_b \psi_b$)

Equation $(*)$ makes no assumption regarding the strength of $H'(t)$. Matrix elements H'_{ab} are functions of time, but we assume we can compute them for any t.

Now let us consider the dynamics of our two-level atom under the specific form of H':

$$H'(t) = ex\mathcal{E}\cos\omega t$$

Unlike in the case of perturbative treatment, we do not require that $\mathcal{E}$ is weak.

As we found out in the previous lecture on selection rules, the diagonal elements of H', namely

$$H'_{aa} = \langle\psi_a|x|\psi_a\rangle e\mathcal{E}\cos\omega t$$

$$H'_{bb} = \langle\psi_b|x|\psi_b\rangle e\mathcal{E}\cos\omega t,$$

vanish because atomic eigenstates have definite parity (even or odd) and the dipole operator ex is of odd parity.

The off-diagonal matrix elements for bound atomic states are real quantities (or can be made real) and are

$$H'_{ba} = H'_{ab} = \langle\psi_b|x|\psi_a\rangle e\mathcal{E}\cos\omega t$$

We define the Rabi frequency Ω as

$$\Omega = \frac{e\mathcal{E}}{\hbar}\langle\psi_a|x|\psi_b\rangle$$

With that our system of two coupled equations for c_a and c_b reduces to

$$\dot{c}_a = -i\Omega\cos\omega t\, e^{-i\omega_{ba}t}\, c_b$$

$$\dot{c}_b = -i\Omega\cos\omega t\, e^{i\omega_{ba}t}\, c_a$$

We can make use of the relation $\cos\omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$

and rewrite the equations as

$$\dot{c}_a = -\frac{1}{2} i\Omega \left[e^{i(\omega-\omega_{ba})t} + e^{-i(\omega+\omega_{ba})t} \right] c_b$$

$$\dot{c}_b = -\frac{1}{2} i\Omega \left[e^{i(\omega+\omega_{ba})t} + e^{-i(\omega-\omega_{ba})t} \right] c_a$$

Now, this system of two equations cannot be solved easily (analytically). However, if we neglect the rapidly oscillating term $e^{-i(\omega+\omega_{ba})t}$, i.e. if we assume that $|\omega-\omega_{ba}| << \omega+\omega_{ba}$ (this is often called the rotating wave approximation - RWA) then we can solve the resulting equations:

$$\dot{c}_a = -\frac{i\Omega}{2} e^{i(\omega-\omega_{ba})t} c_b$$

$$\dot{c}_b = -\frac{i\Omega}{2} e^{-i(\omega-\omega_{ba})t} c_a$$

To simplify notations we will the so called detuning: $\delta \equiv \omega - \omega_{ba}$

With that we get

$$\dot{c}_a = -\frac{i\Omega}{2} e^{i\delta t} c_b$$

$$\dot{c}_b = -\frac{i\Omega}{2} e^{-i\delta t} c_a$$

From the last equation we have $c_a = \frac{2i}{\Omega} e^{i\delta t} \dot{c}_b$, or if we differentiate it with respect to t,

$$\dot{c}_a = -\frac{2\delta}{\Omega} e^{i\delta t} \dot{c}_b + \frac{2i}{\Omega} e^{i\delta t} \ddot{c}_b$$

When we substitute it in the equation for $\dot{c}_a$ we obtain

$$\ddot{c}_b + i\delta \dot{c}_b + \frac{\Omega^2}{4} c_b = 0$$

This is an ordinary differential equation of second order that we know how to solve:

$$\lambda^2 + i\delta \lambda + \frac{\Omega^2}{4} = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left[-i\delta \pm \sqrt{-\delta^2 - \Omega^2}\right] = i\left[-\frac{\delta}{2} \pm \omega_R\right]$$

where $\omega_R \equiv \frac{1}{2}\sqrt{\delta^2 + \Omega^2} = \frac{1}{2}\sqrt{(\omega - \omega_{ba})^2 + \Omega^2}$ – Rabi flopping frequency

$$c_b(t) = A e^{i(-\frac{\delta}{2} + \omega_R)t} + B e^{i(-\frac{\delta}{2} - \omega_R)t} = e^{\frac{i\delta t}{2}}\left[A e^{i\omega_R t} + B e^{-i\omega_R t}\right]$$

Alternatively, we can write it as

$$c_b(t) = e^{-\frac{i\delta t}{2}}\left[F \cos \omega_R t + G \sin \omega_R t\right]$$

Now let us assume that at $t = 0$ the atom was in state a, i.e. $c_a(0) = 1$ $c_b(0) = 0$. Because of the latter we can conclude that $F = 0$ and

$$c_b(t) = G e^{-\frac{i\delta t}{2}} \sin \omega_R t$$

To determine coefficient G we differentiate this expression:

$$\dot{c}_b = G\left[-i\frac{\delta}{2} e^{-\frac{i\delta t}{2}} \sin \omega_R t + \omega_R e^{-\frac{i\delta t}{2}} \cos \omega_R t\right]$$

then

$$c_a(t) = \frac{2i}{\Omega} e^{i\delta t} \dot{c}_b = \frac{2i}{\Omega} G e^{\frac{i\delta t}{2}}\left[-\frac{i\delta}{2} \sin \omega_R t + \omega_R \cos \omega_R t\right]$$

Since $c_a(0) = 1$ we get

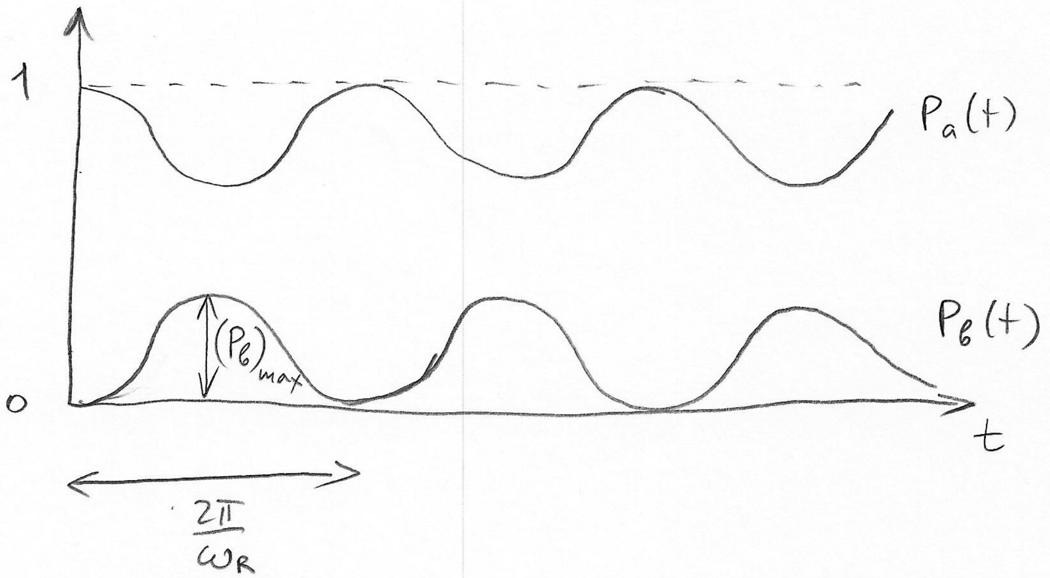
$$1 = \frac{2i}{\Omega} G \omega_R \quad \text{or} \quad G = -\frac{i\Omega}{2\omega_R}$$

So

$$c_b(t) = -\frac{i\Omega}{2\omega_R} e^{-\frac{i\delta t}{2}} \sin \omega_R t$$

and the corresponding probability of $a \to b$ transition is :

$$P_b(t) = |c_b(t)|^2 = \left(\frac{\Omega}{2\omega_R}\right)^2 \sin^2 \omega_R t = \underbrace{\frac{\Omega^2}{\delta^2 + \Omega^2}}_{(P_b)_{max}} \sin^2 \omega_R t$$



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