Name: ______________________

PHYS 452 - Quantum Mechanics II (Spring 2015)
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Midterm Exam 2

Instructions:

- All problems are worth the same number of points (although some might be more difficult than the others). The problem for which you get the lowest score will be dropped. Hence, even if you do not solve one of the problems you can still get the maximum score for the exam.
- This is a closed book exam. No notes, books, phones, tablets, calculators, etc. are allowed. Some information and formulae that might be useful are attached.
- No communication with classmates is allowed during the exam.
- Show all your work, explain your reasoning. Answers without explanations will receive no credit (not even partial one).
- Write legibly. If I cannot read and understand it then I will not be able to grade it.
- Make sure pages are stapled together before submitting your work.

Problem 1. Consider a 1D quantum harmonic oscillator with mass m and frequency ω. Initially, at time $t = 0$, the oscillator is in the ground state. At time $t > 0$ a parametric interaction is turned on. It changes the Hamiltonian to

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2[1 + \gamma\cos(2\omega t)]}{2}$$

with $\gamma \ll 1$.

(a) What is the probability of finding the system in state $|1\rangle$ at time $t > 0$?

(b) What is the probability of finding the system in state $|2\rangle$ at time $t > 0$?

(c) What is the probability of finding the system in state $|4\rangle$ at time $t > 0$?

Problem 2. A particle is in the ground state of a very narrow 1D potential well, $V(x) = -\alpha\delta(x)$, where α is a positive constant. At some moment of time, the strength of the potential is instantaneously changed from α to β. What is the probability that the particle flies away (i.e. leaves the well) after that?

Hint: In this problem you need to know or solve for the ground state of the Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\delta(x)$. This can be done easily if you rememeber that that the wave function must be continuous. The first derivative of the wave function, however, may have a discontinuity at the points of singularity. To determine the magnitude of the jump of the first derivative, you may integrate the Schrödinger equation over an infinitely small region around the point of singularity (i.e. from $-\epsilon$ to ϵ).

Problem 3. Consider scattering from the following spherically symmetric potential:

$$V(r) = \alpha\delta(r - R),$$

where α and R are constants. Assuming that $V(r)$ is weak in some sense

(a) Compute the scattering amplitude

(b) In the low-energy limit, find the expressions for the scattering amplitude, differential and total cross section

Problem 4. Using the Born approximation consider scattering from two identical scattering centers, $U_2(\mathbf{r}) = U_1(\mathbf{r}) + U_1(\mathbf{r} - \mathbf{a})$, separated by some distance $\mathbf{a}$. What is the relation between the differential cross section $\frac{d\sigma_2}{d\Omega}$ and $\frac{d\sigma_1}{d\Omega}$ corresponding to the scattering from U_2 and just a single U_1 when

(a) $ka \ll 1$ and kR is arbitrary (R is the effective range of U_1)

(b) $kR \sim 1$ while $a \gg R$ (i.e. the distance between the two centers is much larger than their effective range)

Appendix: formula sheet

The Schrödinger equation

Time-dependent: $i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$ Stationary: $\hat{H}\psi_n = E_n\psi_n$

De Broglie relations

$\lambda = h/p,\ \nu = E/h$ or $\mathbf{p} = \hbar\mathbf{k},\ E = \hbar\omega$

Heisenberg uncertainty principle

Position-momentum: $\Delta x\,\Delta p_x \geq \frac{\hbar}{2}$ Energy-time: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$ General: $\Delta A\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

Probability current

1D: $j(x,t) = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$ 3D: $j(\mathbf{r},t) = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right)$

Time-evolution of the expectation value of an observable Q (generalized Ehrenfest theorem)

$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \langle\frac{\partial\hat{Q}}{\partial t}\rangle$

Infinite square well ($0 \leq x \leq a$)

Energy levels: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2},\ \ n = 1, 2, ..., \infty$

Eigenfunctions: $\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$ $(0 \leq x \leq a)$

Matrix elements of the position: $\int\limits_0^a \phi_n^*(x)x\,\phi_k(x)dx = \begin{cases} a/2, & n = k \\ 0, & n \neq k;\ n \pm k \text{ is even} \\ -\frac{8nka}{\pi^2(n^2-k^2)^2}, & n \neq k;\ n \pm k \text{ is odd} \end{cases}$

Quantum harmonic oscillator

The few first wave functions ($\alpha = \frac{m\omega}{\hbar}$):

$\phi_0(x) = \frac{\alpha^{1/4}}{\pi^{1/4}}\,e^{-\alpha x^2/2},\ \ \phi_1(x) = \sqrt{2}\frac{\alpha^{3/4}}{\pi^{1/4}}\,x\,e^{-\alpha x^2/2},\ \ \phi_2(x) = \frac{1}{\sqrt{2}}\frac{\alpha^{1/4}}{\pi^{1/4}}\left(2\alpha x^2 - 1\right)e^{-\alpha x^2/2}$

Matrix elements of the position: $\langle\phi_n|\hat{x}|\phi_k\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

$\langle\phi_n|\hat{x}^2|\phi_k\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{k(k-1)}\,\delta_{n,k-2} + \sqrt{(k+1)(k+2)}\,\delta_{n,k+2} + (2k+1)\,\delta_{nk}\right)$

Matrix elements of the momentum: $\langle\phi_n|\hat{p}|\phi_k\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{k}\,\delta_{n,k-1} + \sqrt{n}\,\delta_{k,n-1}\right)$

Equation for the radial component of the wave function of a particle moving in a spherically symmetric potential $V(r)$

$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]R_{nl} = E_{nl}R_{nl}$

Energy levels of the hydrogen atom

$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2},$

The few first radial wave functions for the hydrogen atom ($a = \frac{4\pi\epsilon_0\hbar^2}{me^2}$)

$R_{10} = 2a^{-3/2}\,e^{-\frac{r}{a}}$ $R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1 - \frac{1}{2}\frac{r}{a}\right)e^{-\frac{r}{2a}}$ $R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\,\frac{r}{a}\,e^{-\frac{r}{2a}}$

The few first spherical harmonics

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$ $Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$ $Y_1^{\mp 1} = \pm\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{\mp i\phi} = \pm\sqrt{\frac{3}{8\pi}}\,\frac{x \mp iy}{r}$

Operators of the square of the orbital angular momentum and its projection on the z-axis in spherical coordinates

$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$ $\qquad$ $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

Fundamental commutation relations for the components of angular momentum

$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$ $\qquad$ $[\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x$ $\qquad$ $[\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$

Raising and lowering operators for the z-projection of the angular momentum

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ $\qquad$ Action: $\hat{J}_\pm|j,m\rangle = \sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle$

Relation between coupled and uncoupled representations of states formed by two subsystems with angular momenta j_1 and j_2

$$|J\,M\,j_1\,j_2\rangle = \sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\langle j_1\,m_1\,j_2\,m_2|J\,M\,j_1\,j_2\rangle\,|j_1\,m_1\rangle\,|j_2\,m_2\rangle \qquad m_1+m_2=M$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Stationary perturbation theory formulae

$$H = H^0 + \lambda H', \qquad E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots, \qquad \psi_n = \psi_n^{(0)} + \lambda\psi_n^{(1)} + \lambda^2\psi_n^{(2)} + \ldots$$

$$E_n^{(1)} = H'_{nn}$$

$$\psi_n^{(1)} = \sum_m c_{nm}\psi_m^{(0)}, \quad c_{nm} = \begin{cases} \frac{H'_{mn}}{E_n^{(0)}-E_m^{(0)}}, & n \neq m \\ 0, & n = m \end{cases}$$

$$E_n^{(2)} = \sum_{m\neq n}\frac{|H'_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$\psi_n^{(2)} = \sum_m d_{nm}\psi_m^{(0)}, \quad d_{nm} = \begin{cases} \frac{1}{E_n^{(0)}-E_m^{(0)}}\left(\sum_{k\neq n}\frac{H'_{mk}H'_{kn}}{E_n^{(0)}-E_k^{(0)}}\right) - \frac{H'_{nn}H'_{mn}}{\left(E_n^{(0)}-E_m^{(0)}\right)^2}, & n \neq m \\ 0, & n = m \end{cases}$$

Bohr-Sommerfeld quantization rules

$\int_a^b p(x)dx = (n-\frac{1}{2})\pi\hbar \qquad n = 1, 2, \ldots$

where a and b are classical turning points and $p(x) = \sqrt{2m(E-V(x))}$

If the potential has vertical walls on one or both sides then the above equation becomes

$\int_a^b p(x)dx = (n-\frac{1}{4})\pi\hbar$ or $\int_a^b p(x)dx = n\pi\hbar$ respectively.

Semiclassical barrier tunneling

$$T = \exp\left[-2\int_a^b \kappa(x)dx\right] \qquad \kappa(x) = \frac{1}{\hbar}\sqrt{2m(V(x)-E)}$$

Time-dependence of the wave function

$$H(\mathbf{r},t) = H^0(\mathbf{r}) + \lambda H'(\mathbf{r},t), \qquad H^0\varphi_n = E_n^{(0)}\varphi_n, \qquad \psi(\mathbf{r},t) = \sum_n c_n(t)\varphi_n(\mathbf{r})e^{\frac{-iE_n^{(0)}t}{\hbar}},$$

$$i\hbar\frac{dc_n(t)}{dt} = \lambda\sum_k H'_{nk}e^{i\omega_{nk}t}c_k(t), \qquad H'_{nk} = \langle\phi_n|H'|\phi_k\rangle, \qquad \omega_{nk} = \frac{E_n^{(0)}-E_k^{(0)}}{\hbar}$$

Time-dependent perturbation theory formulae

If $c_n(t_0) = \delta_{nm}$ (e.g. $\psi(\mathbf{r}, t_0) = \varphi_m(\mathbf{r})$, where φ_m is an egenfunction of H^0) and $\lambda H'$ is small then at time $t > t_0$

$c_n(t) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} + \ldots$

where

$c_n^{(0)} = \delta_{nm}, \quad c_n^{(1)}(t) = \frac{1}{i\hbar} \int\limits_{t_0}^{t} H'_{nm}(t')\, e^{i\omega_{nm}t'} dt',$

$c_n^{(2)}(t) = \left(\frac{1}{i\hbar}\right)^2 \sum\limits_k \int\limits_{t_0}^{t} dt' \int\limits_{t_0}^{t'} H'_{nk}(t') H'_{km}(t'')\, e^{i\omega_{nk}t'} e^{i\omega_{km}t''} dt'', \quad \ldots$

Fermi's golden rule

Transition rate: $\Gamma_{i\to f} = \frac{2\pi}{\hbar} |H'_{fi}|^2\, g(E_f)$, Transition probability: $P_{i\to f}(t) = \frac{2\pi t}{\hbar} |H'_{fi}|^2\, g(E_f)$

Stationary quantum scattering

Wave function at $r \to \infty$: $\psi(r,\theta,\phi) \approx A\left[e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}\right], \quad k = \frac{\sqrt{2mE}}{\hbar}$

Differential cross section: $\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$ Total cross section: $\sigma_{\text{tot}} = \int \frac{d\sigma}{d\Omega} d\Omega$

Partial wave analysis

For a spherically symmetric potentials $\psi(r,\theta) = A\left[e^{ikz} + k\sum\limits_{l=0}^{\infty} i^{l+1}(2l+1)a_l\, h_l^{(1)}(kr) P_l(\cos\theta)\right]$

$f(\theta) = \sum\limits_{l=0}^{\infty}(2l+1)a_l\, P_l(\cos\theta) = \frac{1}{k}\sum\limits_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l\, P_l(\cos\theta)$

$\sigma_{\text{tot}} = 4\pi\sum\limits_{l=0}^{\infty}(2l+1)|a_l|^2 = \frac{4\pi}{k^2}\sum\limits_{l=0}^{\infty}(2l+1)\sin^2\delta_l$

Relation between partial wave amplitudes and phase shifts: $a_l = \frac{1}{k}e^{i\delta_l}\sin\delta_l$

Rayleigh formula for a plane wave expansion: $e^{ikz} = \sum\limits_{l=0}^{\infty} i^l(2l+1)j_l(kr)P_l(\cos\theta)$

Lippmann-Schwinger equation

$\psi(\mathbf{r}) = \varphi(\mathbf{r}) + \frac{2m}{\hbar^2}\int G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}',$

where $\varphi(\mathbf{r})$ is the free-particle solution (incident plane wave)

and $G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$ is the Green's function

Born approximation

$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\mathbf{q}\cdot\mathbf{r}'}V(\mathbf{r}')d\mathbf{r}', \quad \mathbf{q} = \mathbf{k}' - \mathbf{k}, \quad q = 2k\sin\frac{\theta}{2}, \quad \mathbf{k} = k\hat{\mathbf{r}}, \quad \mathbf{k}' = k\hat{\mathbf{z}}$

For spherically symmetric potentials $f(\theta) = -\frac{2m}{\hbar^2 q}\int\limits_0^{\infty} rV(r)\sin(qr)dr$

Legendre polynomials

$P_0(x) = 1, \; P_1(x) = x, \; P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \; P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \; \ldots, \; P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$

Orthogonality: $\int\limits_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{ll'}$

Spherical Bessel, Neumann, and Hankel functions

$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}, \quad \ldots, \quad j_l(x) = (-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x}$

$n_0(x) = -\frac{\cos x}{x}, \quad n_1(x) = -\frac{\cos x}{x^2} - \frac{\sin x}{x}, \quad \ldots, \quad n_l(x) = -(-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\cos x}{x}$

$h_l^{(1,2)}(x) = j_l(x) \pm i n_l(x)$

$h_0^{(1)}(x) = -i\frac{e^{ix}}{x}, \quad h_1^{(1)}(x) = \left(-\frac{i}{x^2} - \frac{1}{x}\right)e^{ix}, \quad h_2^{(1)}(x) = \left(-\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right)e^{ix}, \quad \ldots$

$h_0^{(2)}(x) = i\frac{e^{-ix}}{x}, \quad h_1^{(2)}(x) = \left(\frac{i}{x^2} - \frac{1}{x}\right)e^{-ix}, \quad h_2^{(2)}(x) = \left(\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right)e^{-ix}, \quad \ldots$

For $x \ll 1$: $j_l(x) \to \frac{2^l l!}{(2l+1)!}x^l, \quad n_l \to -\frac{(2l)!}{2^l l!}x^{-l-1}$

For $x \gg 1$: $h_l^{(1)} \to \frac{1}{x}(-i)^{l+1}e^{ix}, \quad h_l^{(2)} \to \frac{1}{x}(i)^{l+1}e^{-ix}$

Dirac delta function

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \qquad \delta(x) = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}dk \qquad \delta(-x) = \delta(x) \qquad \delta(cx) = \tfrac{1}{|c|}\delta(x)$$

Fourier transform conventions

$$\tilde{f}(k) = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx \qquad f(x) = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{f}(k)e^{ikx}dk$$

Useful integrals

$$\int_0^{\infty} x^{2k}e^{-\beta x^2}dx = \sqrt{\pi}\tfrac{(2k)!}{k!\,2^{2k+1}\beta^{k+1/2}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int_0^{\infty} x^{2k+1}e^{-\beta x^2}dx = \tfrac{1}{2}\tfrac{k!}{\beta^{k+1}} \quad (\mathrm{Re}\,\beta > 0,\ k = 0,1,2,...)$$

$$\int_0^{\infty} x^{k}e^{-\gamma x}dx = \tfrac{k!}{\gamma^{k+1}} \quad (\mathrm{Re}\,\gamma > 0,\ k = 0,1,2,...)$$

$$\int_{-\infty}^{\infty} e^{-\beta x^2}e^{iqx}dx = \sqrt{\tfrac{\pi}{\beta}}e^{-\frac{q^2}{4\beta}} \quad (\mathrm{Re}\,\beta > 0)$$

$$\int_0^{\pi} \sin^{2k} x\,dx = \pi\tfrac{(2k-1)!!}{2^k\,k!} \quad (k = 0,1,2,...)$$

$$\int_0^{\pi} \sin^{2k+1} x\,dx = \tfrac{2^{k+1}\,k!}{(2k+1)!!} \quad (k = 0,1,2,...)$$

Useful trigonometric identities

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \qquad \cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \qquad \cos\alpha\sin\beta = \tfrac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$