(1) If we know that the orbital angular momentum is zero the quantization rule for the radial motion can be written as

$$\int_0^b p(r)\,dr = \left(n - \frac{1}{2}\right)\pi\hbar \qquad p = \sqrt{2m(E - V(x))}$$

where b is a classical turning point. Substituting $V = -V_0 e^{-r/a}$ into the integral we have

$$\int_0^b \sqrt{2m\left(E + V_0 e^{-r/a}\right)}\,dr = \left(n - \frac{1}{2}\right)\pi\hbar$$

and for a bound state $(E<0)$ $\quad b = a \ln \frac{V_0}{|E|}$

If V_0 is finite in magnitude and there is one and only one bound state that is barely bound (i.e. $E \to 0$) the integrand can be simplified and we get

$$\sqrt{2mV_0} \int_0^{a \ln \frac{V_0}{|E|}} e^{-\frac{r}{2a}}\,dr \approx \left(n - \frac{1}{2}\right)\pi\hbar$$

or

$$\sqrt{2mV_0}\, 2a\left(1 - \sqrt{\frac{|E|}{V_0}}\right) \approx \left(n - \frac{1}{2}\right)\pi\hbar$$

or

$$-E \approx \left[1 - \frac{\left(n - \frac{1}{2}\right)\pi\hbar}{2a\sqrt{2mV_0}}\right]^2 V_0$$

For the disappearance of bound state n (in our case $n=1$) we need to require that $\frac{\left(n-\frac{1}{2}\right)\pi\hbar}{2a\sqrt{2mV_0}}$ becomes larger than 1 (just a litte bit).

$$\frac{\pi\hbar}{4a\sqrt{2mV_0}} \gtrsim 1 \qquad \text{or} \qquad V_0 \lesssim \frac{\pi^2\hbar^2}{32a^2 m}$$

(2) The time-dependent Hamiltonian for our system is :

$$H(x,t) = H_0(x) + V(t)$$

where H_0 is the Hamiltonian of a particle in the box $(V=0)$. Note that $V \neq V(x)$. The time-dependent Schrödinger equation in basis $|\phi_n\rangle$ (ϕ_n are the eigenstates of H_0) is

$$i\hbar \frac{dc_n(t)}{dt} = \sum_k{}' \langle \phi_n | V(t) | \phi_k \rangle e^{i\omega_{nk} t} c_k(t) \qquad \omega_{nk} = \frac{E_n^{(0)} - E_k^{(0)}}{\hbar}$$

In our case $\langle \phi_n | V(t) | \phi_k \rangle = V(t)\, \delta_{nk}$, so

$$\frac{dc_n}{dt} = \frac{1}{i\hbar} V(t)\, c_n \quad \text{and} \quad c_n(t) = c_n(0) \underbrace{e^{-\frac{i}{\hbar}\int_0^t V(t')\,dt'}}_{\text{phase factor}}$$

It is clear that at any moment of time, including $t > T$, the magnitude of coefficiens $c_n(t)$ remains unchanged. Therefore probabilities of all transitions will be zero.

(3) The Hamiltonian of the particle at $t<0$ is

$$H_0 = -\vec{\mu}\cdot\vec{B} = -g\vec{B}\cdot\vec{S} = -gBS_y = -\frac{g\hbar B}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

The eigenvalues and normalized eigenvectors of H_0 are:

$$E_- = g\hbar B \quad \psi_- = \begin{pmatrix} -\frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \qquad E_0 = 0 \quad \psi_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \qquad E_+ = -g\hbar B \quad \psi_+ = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}$$

$m_x = -\hbar \qquad m_x = 0 \qquad m_x = +\hbar$

The perturbing Hamiltonian is

$$H' = -gB_zS_z = -g\beta\delta(t)S_z = -g\hbar\beta\delta(t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

At $t<0$ the particle is in state ψ_+, i.e.

$$C_-(t<0) = 0 \qquad C_0(t<0) = 0 \qquad C_+(t<0) = 1$$

In the framework of the time-dependent perturbation theory the first order correction to C_- (the state with a "flipped spin) is:

$$C_-^{(1)}(t>0) = C_-^{(1)}(t=+\infty) = \frac{1}{i\hbar}\int_{-\infty}^{+\infty} \langle \psi_- | H' | \psi_+ \rangle e^{-i\omega_{-+}t'}\,dt'$$

where $\omega_{-+} = \dfrac{E_- - E_+}{\hbar} = 2g\hbar B$

$$\langle \psi_- | H' | \psi_+ \rangle = -g\hbar\beta\delta(t)\left(-\tfrac{1}{2}\ \tfrac{-i}{\sqrt{2}}\ \tfrac{1}{2}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix} = 0$$

Hence

$$C_-^{(1)}(t>0) = 0$$

and

$$P_-^{(1)}(t>0) = 0$$

The spin will not flip no matter how big β is.

4) Since $E \gg |U(\vec{r})|$ we can use the Born approximation. The scattering amplitude is given by:

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}'} V(\vec{r}')\, d\vec{r}' \qquad q = 2k\sin\frac{\theta}{2}$$

For the first scattering center we have

$$f_1 = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}'} U(\vec{r}')\, d\vec{r}'$$

Scattering from the two centers yields

$$f_2 = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}'}\left[U(\vec{r}') + U(\vec{r}'-\vec{a})\right] d\vec{r}' = f_1 - \frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}'} U(\vec{r}'-\vec{a})\, d\vec{r}' =$$

$$= f_1 - \frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot(\vec{r}''+\vec{a})} U(\vec{r}'')\, d\vec{r}'' = f_1 + e^{i\vec{q}\cdot\vec{a}} f_1 = f_1\left(1 + e^{i\vec{q}\cdot\vec{a}}\right)$$

a) Since $q^2 = 2k^2(1-\cos\theta)$ $\quad qa \ll 1$ when $ka \ll 1$

The scattering amplitude is then

$$f_2 = f_1(1 + 1 + \dots) \approx 2f_1$$

and the differential cross section is

$$\frac{d\sigma_2}{d\Omega} = |f_2|^2 \approx 4|f_1|^2$$

So

$$\frac{d\sigma_2}{d\Omega} \approx 4\frac{d\sigma_1}{d\Omega} \quad \text{and} \quad \sigma_2 \approx 4\sigma_1 \text{ for the total cross section}$$

b) $ka \gg 1$ when $kR \sim 1$ and $a \gg R$

The term $e^{i\vec{q}\cdot\vec{a}}$ oscillates rapidly upon slight changes in θ (and $\vec{q}$)

$$\frac{d\sigma_2}{d\Omega} = |f_2|^2 = |f_1|^2\left(1+e^{-i\vec{q}\cdot\vec{a}}\right)\left(1+e^{i\vec{q}\cdot\vec{a}}\right) = 2|f_1|^2\left(1+\cos\vec{q}\cdot\vec{a}\right) =$$

$$= 2\frac{d\sigma_1}{d\Omega}\left(1+\cos\vec{q}\cdot\vec{a}\right)$$

For the total cross section the last term $(\cos\vec{q}\cdot\vec{a})$ will not contribute because of rapid oscillations and $\sigma_2 = 2\sigma_1$