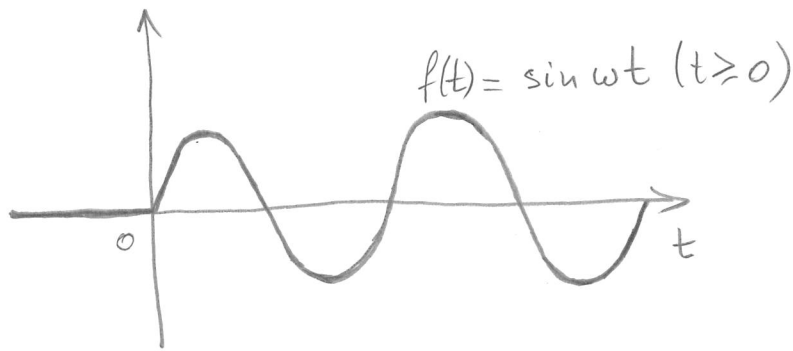


Harmonic perturbation

As an application of time-dependent perturbation theory let us consider a perturbation that is turned on at $t=0$ and is subsequently monochromatic in time. The perturbation acts on a system whose Hamiltonian is H^0 . Such a model is relevant for describing an atom that interacts with a weak electromagnetic field. The explicit formula for the perturbation is



with a weak electromagnetic field. The explicit formula for the perturbation is

$$H'(\vec{r}, t) = \begin{cases} 0, & t < 0 \\ H'(\vec{r}) \sin \omega t, & t \geq 0 \end{cases}$$

Using the expression for the probability amplitudes obtained in the previous lecture, namely

$$c_{fi}^{(1)} = \frac{H'_{fi}}{i\hbar} \int_{t_0}^t e^{i\omega_{fi}t'} f(t') dt'$$

we get (we will drop the (1) superscript for simplicity)

$$c_{fi}(t) = \frac{H'_{fi}}{i\hbar} \int_0^t e^{i\omega_{fi}t'} \left(\frac{e^{i\omega t'} - e^{-i\omega t'}}{2i} \right) dt' =$$

$$= -\frac{H'_{fi}}{2\hbar i} \left[\frac{e^{i(\omega_{fi}+\omega)t} - 1}{\omega_{fi} + \omega} - \frac{e^{i(\omega_{fi}-\omega)t} - 1}{\omega_{fi} - \omega} \right]$$

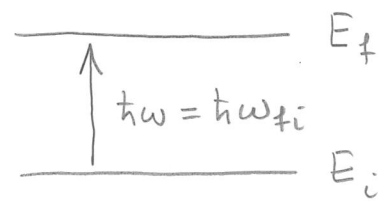
If we use the relation $e^{i\alpha} - 1 = e^{i\frac{\alpha}{2}} (e^{i\frac{\alpha}{2}} - e^{-i\frac{\alpha}{2}}) = 2ie^{i\frac{\alpha}{2}} \sin \frac{\alpha}{2}$

we can rewrite it as

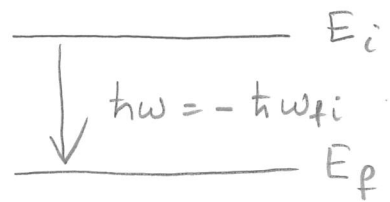
$$c_{fi}(t) = -\frac{H'_{fi}}{\hbar} \left[\frac{e^{i(\omega_{fi}+\omega)t/2} \sin \frac{(\omega_{fi}+\omega)t}{2}}{\omega_{fi} + \omega} - \frac{e^{i(\omega_{fi}-\omega)t/2} \sin \frac{(\omega_{fi}-\omega)t}{2}}{\omega_{fi} - \omega} \right]$$

c_{fi} becomes large when $\omega \approx \pm \omega_{fi}$ (resonant frequency)

In the case $\omega \approx +\omega_{fi}$ $E_f > E_i$ and the system absorbs energy and "jumps" to a higher energy level $E_f = E_i + \hbar\omega$



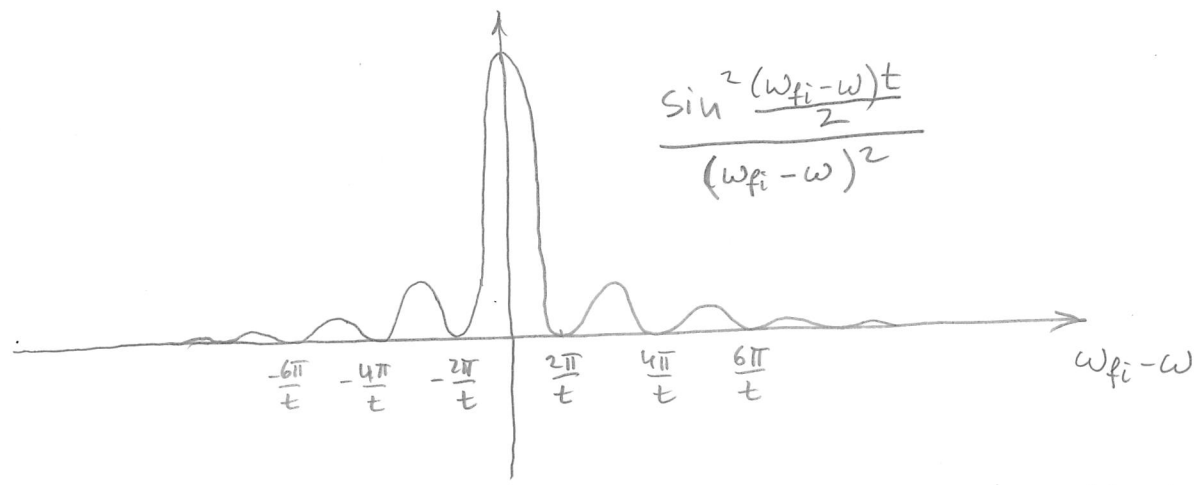
In the case $\omega \approx -\omega_{fi}$ $E_f < E_i$ and we can say that the perturbation induces a decay in energy $E_f = E_i - \hbar\omega$



The decay process is stimulated by a photon of the same frequency in the perturbation field (stimulated emission)

Let us consider the case when $\omega_{fi} > 0$. Under this condition the second term in the expression for $c_{fi}(t)$ above dominates. The corresponding probability of transition is then

$$P_{fi} = |c_{fi}|^2 = \frac{|\mathcal{H}'_{fi}|^2}{\hbar^2(\omega_{fi} - \omega)^2} \sin^2 \left[\frac{(\omega_{fi} - \omega)t}{2} \right]$$



From the sketch above it is evident that the states falling in the interval $|\hbar\omega_{fi} - \hbar\omega| = |E_f - (E_i + \hbar\omega)| \leq \frac{2\pi\hbar}{t} \approx \Delta E$ have the greatest probability of being

excited, after the perturbation has acted for time t . Hence, the above inequality provides the uncertainty in the energy (after time t) that will be observed:

$$\Delta E \approx \frac{\hbar}{t} \quad E_f \approx E_i + t\omega \pm \Delta E$$

Our analysis has yielded the conservation of energy within the uncertainty.

Let us now consider long-time evolution. The expression for P_{fi} in the limit $t \rightarrow \infty$ (or, alternatively, $\omega \rightarrow \infty$) becomes resonant at $\omega = \pm \omega_{fi}$ and, in fact, approaches a delta function:

$$\frac{1}{\pi} \lim_{t \rightarrow \infty} \frac{\sin^2 dt}{d^2 t} = \delta(d) \quad \left(\int_{-\infty}^{+\infty} \frac{\sin^2 dt}{d^2 t} dd = \int_{-\infty}^{+\infty} \frac{\sin^2 dt}{d^2 t^2} dt = \int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2} dx = \pi \right)$$

$$P_{fi} = \frac{|\mathcal{H}'_{fi}|^2 \frac{t}{2}}{t^2 (\omega_{fi} \mp \omega)^2 \frac{t}{2}} \sin^2 \left[\frac{(\omega_{fi} \mp \omega)t}{2} \right] \rightarrow \frac{\pi t |\mathcal{H}'_{fi}|^2}{2 t^2} \delta(\omega_{fi} \mp \omega)$$

The corresponding transition probability rate is

$$W_{fi} = \frac{dP_{fi}}{dt} = \frac{\pi}{2t^2} |\mathcal{H}'_{fi}|^2 \delta(\omega_{fi} \mp \omega)$$

In the formulae for W_{fi} and P_{fi} the presence of the delta function constitutes the fact that in the long-time limit the Fourier transform of the perturbation becomes sharply peaked around the frequency of the perturbing field. Thus, the system will only see a single frequency. Since the uncertainty in energy, $\frac{\hbar}{t}$, vanishes in this limit, the argument of the delta function is also a manifestation of the conservation of energy.

In the short-time approximation, when $(\omega_{fi} - \omega)t \ll 1$

the expression for P_{fi} may be expanded into a Taylor series in t :

$$P_{fi} = \frac{t^2 |\mathcal{H}'_{fi}|^2}{4\hbar^2} \quad \text{and then} \quad W_{fi} = \frac{t |\mathcal{H}'_{fi}|^2}{2\hbar^2}$$

At early times the rate at which transitions to the final state occur grows linearly with time.